Sample Term Structure

• We shall construct interest rate trees consistent with the sample term structure in the following table.

• Assume the short rate volatility is such that \( v \equiv \frac{r_h}{r_\ell} = 1.5 \), independent of time.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate (%)</td>
<td>4</td>
<td>4.2</td>
<td>4.3</td>
</tr>
<tr>
<td>One-period forward rate (%)</td>
<td>4</td>
<td>4.4</td>
<td>4.5</td>
</tr>
<tr>
<td>Discount factor</td>
<td>0.96154</td>
<td>0.92101</td>
<td>0.88135</td>
</tr>
</tbody>
</table>
An Approximate Calibration Scheme

- Start with the implied one-period forward rates and then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today’s one-period spot rate.
- In general, let $f_j$ denote the forward rate in period $j$.
- This forward rate can be derived from the market discount function via $f_j = (d(j)/d(j+1)) - 1$ (see Exercise 5.6.3 in text).
An Approximate Calibration Scheme (continued)

- Since the $i$th short rate, $1 \leq i \leq j$, occurs with probability $2^{-(j-1)} \binom{j-1}{i-1}$, this means

$$\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.$$  

- Thus

$$r_j = \left( \frac{2}{1 + v_j} \right)^{j-1} f_j. \quad (78)$$

- The binomial interest rate tree is trivial to set up.
An Approximate Calibration Scheme (concluded)

• The ensuing tree for the sample term structure appears in figure next page.

• For example, the price of the zero-coupon bond paying $1 at the end of the third period is

\[
\frac{1}{4} \times \frac{1}{1.04} \times \left( \frac{1}{1.0352} \times \left( \frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left( \frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)
\]

or 0.88155, which exceeds discount factor 0.88135.

• The tree is thus not calibrated.

• Indeed, this bias is inherent (see text).
Baseline rates

A 4.0%
B 3.52%
C 2.88%
D

B 5.28%
C 4.32%
D

C 6.48%
D

Implied forward rates: 4.0% 4.4% 4.5%

period 1 period 2 period 3
Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the $m$-period zero-coupon bond as computing some function of the unknown baseline rate $r_m$ called $f(r_m)$.
- A root-finding method is applied to solve $f(r_m) = P$ for $r_m$ given the zero’s price $P$ and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- It runs in cubic time, hopelessly slow.
Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in quadratic time by the use of forward induction (Jamshidian, 1991).
- The scheme records how much $1 at a node contributes to the model price.
- This number is called the state price.
  - It is the price of a state contingent claim that pays $1 at that particular node (state) and 0 elsewhere.
- The column of state prices will be established by moving forward from time 1 to time $n$. 
Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time \( j \) and there are \( j + 1 \) nodes.
  - The baseline rate for period \( j \) is \( r \equiv r_j \).
  - The multiplicative ratio be \( v \equiv v_j \).
  - \( P_1, P_2, \ldots, P_j \) are the state prices a period prior, corresponding to rates \( r, rv, \ldots, rv^{j-1} \).

- By definition, \( \sum_{i=1}^{j} P_i \) is the price of the \((j - 1)\)-period zero-coupon bond.
Binomial Interest Rate Tree Calibration (continued)

- One dollar at time $j$ has a known market value of $1/[1 + S(j)]^j$, where $S(j)$ is the $j$-period spot rate.
- Alternatively, this dollar has a present value of

$$g(r) \equiv \frac{P_1}{1 + r} + \frac{P_2}{(1 + rv)} + \frac{P_3}{(1 + rv^2)} + \cdots + \frac{P_j}{(1 + rv^{j-1})}.$$  

- So we solve

$$g(r) = \frac{1}{[1 + S(j)]^j} \quad (79)$$

for $r$. 
Binomial Interest Rate Tree Calibration (continued)

• Given a decreasing market discount function, a unique positive solution for \( r \) is guaranteed.

• The state prices at time \( j \) can now be calculated (see figure (a) next page).

• We call a tree with these state prices a binomial state price tree (see figure (b) next page).

• The calibrated tree is depicted on p. 761.
Implied forward rates: 4.0%  4.4%  4.5%

period 1  period 2  period 3
Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the \( r \) in Eq. (79) on p. 758 as \( g'(r) \) is easy to evaluate.

- The monotonicity and the convexity of \( g(r) \) also facilitate root finding.

- The total running time is \( O(Cn^2) \), where \( C \) is the maximum number of times the root-finding routine iterates, each consuming \( O(n) \) work.

- With a good initial guess, the Newton-Raphson method converges in only a few steps\(^a\)

\(^a\)Lyuu (1999).
A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.

- The baseline rate for the second period, $r_2$, satisfies

$$\frac{0.480769}{1 + r_2} + \frac{0.480769}{1 + 1.5 \times r_2} = 0.92101.$$ 

- The result is $r_2 = 3.526\%$.

- This is used to derive the next column of state prices shown in figure (b) on p. 760 as 0.232197, 0.460505, and 0.228308.

- Their sum gives the correct market discount factor 0.92101.
A Numerical Example (concluded)

- The baseline rate for the third period, \( r_3 \), satisfies
  \[
  \frac{0.232197}{1 + r_3} + \frac{0.460505}{1 + 1.5 \times r_3} + \frac{0.228308}{1 + (1.5)^2 \times r_3} = 0.88135.
  \]

- The result is \( r_3 = 2.895\% \).

- Now, redo the calculation on p. 753 using the new rates:
  \[
  \frac{1}{4} \times \frac{1}{1.04} \times \left[ \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) + \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) \right],
  \]
  which equals 0.88135, an exact match.

- The tree on p. 761 prices without bias the benchmark securities.

- The term structure dynamics is shown on p. 765.
Spread of Nonbenchmark Bonds

- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.

- The incremental return over the benchmark bonds is called spread.

- We look for the spread that, when added uniformly over the short rates in the tree, makes the model price equal the market price.

- We will apply the spread concept to option-free bonds here.
Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 768.
- Consider a security with cash flow $C_i$ at time $i$ for $i = 1, 2, 3$.
- Its model price is $p(s)$, which is equal to

$$
\frac{1}{1.04 + s} \times \left[ C_1 + \frac{1}{2} \times \frac{1}{1.03526 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.02895 + s} + \frac{C_3}{1.04343 + s} \right) \right) + \\
\frac{1}{2} \times \frac{1}{1.05289 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.04343 + s} + \frac{C_3}{1.06514 + s} \right) \right) \right].
$$

- Given a market price of $P$, the spread is the $s$ that solves $P = p(s)$. 

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Implied forward rates: 4.0% 4.4% 4.5%

<table>
<thead>
<tr>
<th>Period</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A 4.0%</td>
</tr>
</tbody>
</table>
| 2      | B 5.289%+
| 3      | C 6.514%+

Diagram:

A -> B -> C -> D

A 4.0%+
B 3.526%+
C 2.895%+
D

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Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of $s$.
- We will employ the Newton-Raphson root-finding method to solve $p(s) - P = 0$ for $s$.
- But a quick look at the equation above reveals that evaluating $p'(s)$ directly is infeasible.
- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.
Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate $r$.

- In the process of computing the model price $p(s)$, a price $p_A(s)$ is computed at A.

- Prices computed at A’s two successor nodes B and C are discounted by $r + s$ to obtain $p_A(s)$ as follows,

$$p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1 + r + s)},$$

where $c$ denotes the cash flow at A.
Spread of Nonbenchmark Bonds (continued)

- To compute $p'_A(s)$ as well, node A calculates

$$p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1 + r + s)} - \frac{p_B(s) + p_C(s)}{2(1 + r + s)^2}. \quad (80)$$

- This is easy if $p'_B(s)$ and $p'_C(s)$ are also computed at nodes B and C.

- Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 772).

- This is called the differential tree method.\(^a\)

\(^a\)Lyuu (1999).
\[ p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1+r+s)} \]
\[ p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1+r+s)} - \frac{p_B(s) + p_C(s)}{2(1+r+s)} \]
Spread of Nonbenchmark Bonds (continued)

- Let $C$ represent the number of times the tree is traversed, which takes $O(n^2)$ time.
- The total running time is $O(Cn^2)$.
- In practice $C$ is a small constant.
- The memory requirement is $O(n)$. 
Spread of Nonbenchmark Bonds (continued)

<table>
<thead>
<tr>
<th>Number of partitions $n$</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
<th>Number of partitions</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>7.850</td>
<td>5</td>
<td>10500</td>
<td>3503.410</td>
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<td>1500</td>
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<td>5</td>
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<td>5</td>
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<td>2500</td>
<td>198.770</td>
<td>5</td>
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<td>5</td>
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<td>7548.760</td>
<td>5</td>
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<td>6500</td>
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<td>5</td>
<td>16500</td>
<td>8502.950</td>
<td>5</td>
</tr>
<tr>
<td>7500</td>
<td>1761.110</td>
<td>5</td>
<td>17500</td>
<td>9523.900</td>
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<tr>
<td>8500</td>
<td>2269.750</td>
<td>5</td>
<td>18500</td>
<td>10617.370</td>
<td>5</td>
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<tr>
<td>9500</td>
<td>2834.170</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

75MHz Sun SPARCstation 20.
Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (p. 776).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread and static spread of the nonbenchmark bond over an otherwise identical benchmark bond.
Cash flows:

5 5 105
More Applications of the Differential Tree: Calibrating Black-Derman-Toy (in seconds)

<table>
<thead>
<tr>
<th>Number of years</th>
<th>Running time</th>
<th>Number of years</th>
<th>Running time</th>
<th>Number of years</th>
<th>Running time</th>
</tr>
</thead>
<tbody>
<tr>
<td>3000</td>
<td>398.880</td>
<td>39000</td>
<td>8562.640</td>
<td>75000</td>
<td>26182.080</td>
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<tr>
<td>6000</td>
<td>1697.680</td>
<td>42000</td>
<td>9579.780</td>
<td>78000</td>
<td>28138.140</td>
</tr>
<tr>
<td>9000</td>
<td>2539.040</td>
<td>45000</td>
<td>10785.850</td>
<td>81000</td>
<td>30230.260</td>
</tr>
<tr>
<td>12000</td>
<td>2803.890</td>
<td>48000</td>
<td>11905.290</td>
<td>84000</td>
<td>32317.050</td>
</tr>
<tr>
<td>15000</td>
<td>3149.330</td>
<td>51000</td>
<td>13199.470</td>
<td>87000</td>
<td>34487.320</td>
</tr>
<tr>
<td>18000</td>
<td>3549.100</td>
<td>54000</td>
<td>14411.790</td>
<td>90000</td>
<td>36795.430</td>
</tr>
<tr>
<td>21000</td>
<td>3990.050</td>
<td>57000</td>
<td>15932.370</td>
<td>120000</td>
<td>63767.690</td>
</tr>
<tr>
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</tr>
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<td>27000</td>
<td>5211.830</td>
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<td>19037.910</td>
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<tr>
<td>36000</td>
<td>7611.630</td>
<td>72000</td>
<td>24292.740</td>
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<td>313480.390</td>
</tr>
</tbody>
</table>

75MHz Sun SPARCstation 20, one period per year.
More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)

<table>
<thead>
<tr>
<th>Number of partitions</th>
<th>American call</th>
<th>Number of partitions</th>
<th>American put</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Running time</td>
<td>Number of iterations</td>
<td>Running time</td>
</tr>
<tr>
<td>100</td>
<td>0.008210</td>
<td>2</td>
<td>100</td>
</tr>
<tr>
<td>200</td>
<td>0.033310</td>
<td>2</td>
<td>200</td>
</tr>
<tr>
<td>300</td>
<td>0.072940</td>
<td>2</td>
<td>300</td>
</tr>
<tr>
<td>400</td>
<td>0.129180</td>
<td>2</td>
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<td>500</td>
<td>0.201850</td>
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<td>500</td>
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<td>600</td>
<td>0.290480</td>
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<td>700</td>
</tr>
<tr>
<td>800</td>
<td>0.522040</td>
<td>2</td>
<td>800</td>
</tr>
</tbody>
</table>

Intel 166MHz Pentium, running on Microsoft Windows 95.
Fixed-Income Options

• Consider a two-year 99 European call on the three-year, 5% Treasury.

• Assume the Treasury pays annual interest.

• From p. 780 the three-year Treasury’s price minus the $5 interest could be $102.046, $100.630, or $98.579 two years from now.

• Since these prices do not include the accrued interest, we should compare the strike price against them.

• The call is therefore in the money in the first two scenarios, with values of $3.046 and $1.630, and out of the money in the third scenario.
Fixed-Income Options (continued)

- The option value is calculated to be $1.458 on p. 780(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only if the Treasury is worth $98.579 without the accrued interest.
- The option value is computed to be $0.096 on p. 780(b).
Fixed-Income Options (concluded)

• The present value of the strike price is 
  \[ PV(X) = 99 \times 0.92101 = 91.18. \]

• The Treasury is worth \( B = 101.955 \).

• The present value of the interest payments during the life of the options is
  \[ PV(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275. \]

• The call and the put are worth \( C = 1.458 \) and \( P = 0.096 \), respectively.

• Hence the put-call parity is preserved:
  \[ C = P + B - PV(I) - PV(X). \]
Delta or Hedge Ratio

• How much does the option price change in response to changes in the price of the underlying bond?

• This relation is called delta (or hedge ratio) defined as

\[
\frac{O_h - O_\ell}{P_h - P_\ell}.
\]

• In the above \( P_h \) and \( P_\ell \) denote the bond prices if the short rate moves up and down, respectively.

• Similarly, \( O_h \) and \( O_\ell \) denote the option values if the short rate moves up and down, respectively.
Delta or Hedge Ratio (concluded)

- Since delta measures the sensitivity of the option value to changes in the underlying bond price, it shows how to hedge one with the other.

- Take the call and put on p. 780 as examples.

- Their deltas are

\[
\frac{0.774 - 2.258}{99.350 - 102.716} = 0.441, \\
\frac{0.200 - 0.000}{99.350 - 102.716} = -0.059,
\]

respectively.
Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an $n$-period zero-coupon bond.
- First find its yield to maturity $y_h$ ($y_\ell$, respectively) at the end of the initial period if the rate rises (declines, respectively).
- The yield volatility for our model is defined as $(1/2) \ln(y_h/y_\ell)$. 
Volatility Term Structures (continued)

• For example, based on the tree on p. 761, the two-year zero’s yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.

• Its yield volatility is therefore

\[
\frac{1}{2} \ln \left( \frac{0.05289}{0.03526} \right) = 20.273\%.
\]
Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the rate rises, the price of the zero one year from now will be
  \[ \frac{1}{2} \times \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096. \]
- Thus its yield is \( \sqrt{\frac{1}{0.90096}} - 1 = 0.053531. \)
- If the rate declines, the price of the zero one year from now will be
  \[ \frac{1}{2} \times \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225. \]
Volatility Term Structures (continued)

- Thus its yield is \( \sqrt{\frac{1}{0.93225}} - 1 = 0.0357 \).
- The yield volatility is hence
  \[
  \frac{1}{2} \ln \left( \frac{0.053531}{0.0357} \right) = 20.256\%
  \]
  slightly less than the one-year yield volatility.
- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.
Spot rate volatility

Short rate volatility given flat %10 volatility term structure.
Volatility Term Structures (continued)

- We started with $v_i$ and then derived the volatility term structure.

- In practice, the steps are reversed.

- The volatility term structure is supplied by the user along with the term structure.

- The $v_i$—hence the short rate volatilities via Eq. (76) on p. 741—and the $r_i$ are then simultaneously determined.

- The result is the Black-Derman-Toy model.
Volatility Term Structures (concluded)

• Suppose the user supplies the volatility term structure which results in \((v_1, v_2, v_3, \ldots)\) for the tree.

• The volatility term structure one period from now will be determined by \((v_2, v_3, v_4, \ldots)\) not \((v_1, v_2, v_3, \ldots)\).

• The volatility term structure supplied by the user is hence not maintained through time.

• This issue will be addressed by other types of (complex) models.
Foundations of Term Structure Modeling
[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.
[The] fixed-income traders I knew seemed smarter than the equity trader [...] there’s no competitive edge to being smart in the equities business.

— Emanuel Derman,

*My Life as a Quant* (2004)
Terminology

• A period denotes a unit of elapsed time.
  – Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.

• Bonds will be assumed to have a par value of one unless stated otherwise.

• The time unit for continuous-time models will usually be measured by the year.
Standard Notations

The following notation will be used throughout.

\( t \): a point in time.

\( r(t) \): the one-period riskless rate prevailing at time \( t \) for repayment one period later (the instantaneous spot rate, or short rate, at time \( t \)).

\( P(t,T) \): the present value at time \( t \) of one dollar at time \( T \).
Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period—in other words, the $(T - t)$-period spot rate at time $t$.

$F(t, T, M)$: the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$. 
Standard Notations (concluded)

\( f(t, T, L) \): the \( L \)-period forward rate at time \( T \) implied at time \( t \) stated on a per-period basis and compounded once per period.

\( f(t, T) \): the one-period or instantaneous forward rate at time \( T \) as seen at time \( t \) stated on a per period basis and compounded once per period.

- It is \( f(t, T, 1) \) in the discrete-time model and \( f(t, T, dt) \) in the continuous-time model.
- Note that \( f(t, t) \) equals the short rate \( r(t) \).
Fundamental Relations

- The price of a zero-coupon bond equals

\[ P(t, T) = \begin{cases} 
(1 + r(t, T))^{-(T-t)}, & \text{in discrete time}, \\
e^{-r(t,T)(T-t)}, & \text{in continuous time}. 
\end{cases} \]

- \( r(t, T) \) as a function of \( T \) defines the spot rate curve at time \( t \).

- By definition,

\[ f(t, t) = \begin{cases} 
r(t, t + 1), & \text{in discrete time}, \\
r(t, t), & \text{in continuous time}. 
\end{cases} \]
Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

\[ F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \quad (81) \]

- The forward price equals the future value at time \( T \) of the underlying asset (see text for proof).

- Equation (81) holds whether the model is discrete-time or continuous-time.
Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

\[
f(t, T, L) = \left( \frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1 \tag{82}
\]

in discrete time.

- \( f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L)} - 1 \right) \) is the analog to Eq. (82) under simple compounding.
Fundamental Relations (continued)

• In continuous time,

\[ f(t, T, L) = -\frac{\ln F(t, T, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L} \]  

(83)

by Eq. (81) on p. 800.

• Furthermore,

\[
\begin{align*}
 f(t, T, \Delta t) &= \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \rightarrow -\frac{\partial \ln P(t, T)}{\partial T} \\
&= -\frac{\partial P(t, T)/\partial T}{P(t, T)}.
\end{align*}
\]
Fundamental Relations (continued)

• So

\[ f(t, T) \equiv \lim_{\Delta t \to 0} f(t, T, \Delta t) = -\frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T. \]  

(84)

• Because Eq. (84) is equivalent to

\[ P(t, T) = e^{-\int_t^T f(t, s) ds}, \]  

(85)

the spot rate curve is

\[ r(t, T) = \frac{1}{T - t} \int_t^T f(t, s) \, ds. \]
Fundamental Relations (concluded)

• The discrete analog to Eq. (85) is

\[ P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}. \]

• The short rate and the market discount function are related by

\[ r(t) = -\frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}. \]
Risk-Neutral Pricing

- Assume the local expectations theory.

- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,
    \[
    E_t\left[\frac{P(t + 1, T)}{P(t, T)}\right] = 1 + r(t). \tag{86}
    \]
  - Relation (86) in fact follows from the risk-neutral valuation principle.$^a$

\[^a\text{Theorem 14 on p. 429.}\]
Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability \( \pi \).

- Rewrite Eq. (86) as

\[
E_t^{\pi}[P(t + 1, T)] \\
1 + r(t)
\]

\[= P(t, T).\]

- It says the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate.
Risk-Neutral Pricing (continued)

• Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ \frac{P(t+1, T)}{1 + r(t)} \right]
= E_t^\pi \left[ \frac{E_{t+1}^\pi [P(t+2, T)]}{(1 + r(t))(1 + r(t+1))} \right] = \ldots
= E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t+1)) \cdots (1 + r(T-1))} \right]. \quad (87)
\]
Risk-Neutral Pricing (concluded)

• Equation (86) on p. 805 can also be expressed as

\[ E_t[P(t + 1, T)] = F(t, t + 1, T). \]

• Hence the forward price for the next period is an unbiased estimator of the expected bond price.
Continuous-Time Risk-Neutral Pricing

• In continuous time, the local expectations theory implies

\[ P(t, T) = E_t \left[ e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \] (88)

• Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which
denotes the rolled-over money market account.

• When the local expectations theory holds, riskless
arbitrage opportunities are impossible.
Interest Rate Swaps

• Consider an interest rate swap made at time \( t \) with payments to be exchanged at times \( t_1, t_2, \ldots, t_n \).

• The fixed rate is \( c \) per annum.

• The floating-rate payments are based on the future annual rates \( f_0, f_1, \ldots, f_{n-1} \) at times \( t_0, t_1, \ldots, t_{n-1} \).

• For simplicity, assume \( t_{i+1} - t_i \) is a fixed constant \( \Delta t \) for all \( i \), and the notional principal is one dollar.

• If \( t < t_0 \), we have a forward interest rate swap.

• The ordinary swap corresponds to \( t = t_0 \).
Interest Rate Swaps (continued)

- The amount to be paid out at time $t_{i+1}$ is $(f_i - c) \Delta t$ for the floating-rate payer.
- Simple rates are adopted here.
- Hence $f_i$ satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$
Interest Rate Swaps (continued)

- The value of the swap at time $t$ is thus

$$\sum_{i=1}^{n} E_t^{\pi} \left[ e^{-\int_{t_{i-1}}^{t_i} r(s) \, ds} (f_{i-1} - c) \Delta t \right]$$

$$= \sum_{i=1}^{n} E_t^{\pi} \left[ e^{-\int_{t_{i-1}}^{t_i} r(s) \, ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right]$$

$$= \sum_{i=1}^{n} \left[ P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \right]$$

$$= P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).$$
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.
Swapping Rate

- The swap rate, which gives the swap zero value, equals

\[ S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i) \Delta t}. \]  \hspace{1cm} (89)

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

- For an ordinary swap, \( P(t, t_0) = 1 \).
The Binomial Model

• The analytical framework can be nicely illustrated with the binomial model.

• Suppose the bond price $P$ can move with probability $q$ to $Pu$ and probability $1-q$ to $Pd$, where $u > d$:

$$P \xrightarrow{1-q} Pd \xleftarrow{q} Pu$$
The Binomial Model (continued)

• Over the period, the bond’s expected rate of return is

\[ \hat{\mu} \equiv \frac{qP_u + (1 - q)P_d}{P} - 1 = qu + (1 - q)d - 1. \]  

(90)

• The variance of that return rate is

\[ \hat{\sigma}^2 \equiv q(1 - q)(u - d)^2. \]  

(91)

• The bond whose maturity is only one period away will move from a price of $1/(1 + r)$ to its par value $1$.

• This is the money market account modeled by the short rate.
The Binomial Model (continued)

• The market price of risk is defined as $\lambda \equiv (\hat{\mu} - r)/\hat{\sigma}$.

• As in the continuous-time case, it can be shown that $\lambda$ is independent of the maturity of the bond (see text).
The Binomial Model (concluded)

- Now change the probability from $q$ to
  
  $$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1 + r) - d}{u - d},$$  
  \hspace{1cm} (92) 
  
  which is independent of bond maturity and $q$.

  - Recall the BOPM.

- The bond’s expected rate of return becomes
  
  $$\frac{pPu + (1 - p)Pd}{P} - 1 = pu + (1 - p)d - 1 = r.$$ 

- The local expectations theory hence holds under the new probability measure $p$. 

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Numerical Examples

• Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

• Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

\[ 4\% \quad 8\% \quad 2\% \]
Numerical Examples (continued)

- No real-world probabilities are specified.
- The prices of one- and two-year zero-coupon bonds are, respectively,
  
  \[ \frac{100}{1.04} = 96.154, \frac{100}{(1.05)^2} = 90.703. \]

- They follow the binomial processes on p. 821.
Numerical Examples (continued)

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then
  \[ (1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%, \]
  where \( p \) denotes the risk-neutral probability of an up move in rates.
Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

• A one-year European call on the two-year zero with a $95 strike price has the payoffs,

  $C = \begin{cases} 0.000 & \text{if } S < 95 \\ 3.039 & \text{if } S \geq 95 \end{cases}$

• To solve for the option value $C$, we replicate the call by a portfolio of $x$ one-year and $y$ two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

• This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

• They give \( x = -0.5167 \) and \( y = 0.5580. \)

• Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options (continued)

• This price is derived without assuming any version of an expectations theory.

• Instead, the arbitrage-free price is derived by replication.

• The price of an interest rate contingent claim does not depend directly on the real-world probabilities.

• The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

• An equivalent method is to utilize risk-neutral pricing.

• The above call option is worth

\[ C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]

the same as before.

• This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.
Numerical Examples: Futures and Forward Prices

• A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:

  $F \leftarrow \begin{align*}
  92 &= 100 - 8 \\
  98 &= 100 - 2
  \end{align*}

• As the futures price $F$ is the expected future payoff (see text), $F = (1 - p) \times 92 + p \times 98 = 93.914$.

• On the other hand, the forward price for a one-year forward contract on a one-year zero-coupon bond equals $90.703/96.154 = 94.331\%$.

• The forward price exceeds the futures price.
Numerical Examples: Mortgage-Backed Securities

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 830.
- Its fair price is
  \[ M = \frac{(1 - p) \times 102.222 + p \times 107.941}{1.04} = 100.045. \]
- Identical results could have been obtained via arbitrage considerations.
The left diagram depicts the cash flow; the right diagram illustrates the price process.
Numerical Examples: MBSs (continued)

• Suppose that the security can be prepaid at par.

• It will be prepaid only when its price is higher than par.

• Prepayment will hence occur only in the “down” state when the security is worth 102.941 (excluding coupon).

• The price therefore follows the process,

\[ M \begin{cases} 102.222 \\ 105 \end{cases} \]

• The security is worth

\[ M = \frac{(1 - p) \times 102.222 + p \times 105}{1.04} = 99.142. \]
Numerical Examples: MBSs (continued)

- The cash flow of the principal-only (PO) strip comes from the mortgage’s principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 833(a)).
- Their prices hence follow the processes on p. 833(b).
- The fair prices are

\[
\begin{align*}
PO &= \frac{(1 - p) \times 92.593 + p \times 100}{1.04} = 91.304, \\
IO &= \frac{(1 - p) \times 9.630 + p \times 5}{1.04} = 7.839.
\end{align*}
\]
The price 9.630 is derived from $5 + \frac{5}{1.08}$. 
Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.

- Let the floater (FLT) receive the one-year rate.

- Then the inverse floater (INV) must have a coupon rate of

\[(10\% - \text{one-year rate})\]

- to make the overall coupon rate 5%.

- Their cash flows as percentages of par and values are shown on p. 835.
Numerical Examples: MBSs (concluded)

- On p. 835, the floater’s price in the up node, 104, is derived from $4 + (108/1.08)$.

- The inverse floater’s price 100.444 is derived from $6 + (102/1.08)$.

- The current prices are

  \[
  \begin{align*}
  \text{FLT} & = \frac{1}{2} \times \frac{104}{1.04} = 50, \\
  \text{INV} & = \frac{1}{2} \times \frac{(1 - p) \times 100.444 + p \times 106}{1.04} = 49.142.
  \end{align*}
  \]
Equilibrium Term Structure Models
8. What’s your problem? Any moron can understand bond pricing models.

— *Top Ten Lies Finance Professors Tell Their Students*
Introduction

- This chapter surveys equilibrium models.
- Since the spot rates satisfy

\[ r(t, T) = -\frac{\ln P(t, T)}{T - t}, \]

the discount function \( P(t, T) \) suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.
The Vasicek Model\textsuperscript{a}

- The short rate follows
  \[ dr = \beta(\mu - r)\, dt + \sigma\, dW. \]

- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).

- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma\, dW \).

- Since the process is an Ornstein-Uhlenbeck process,
  \[ E[r(T) \mid r(t) = r] = \mu + (r - \mu) e^{-\beta(T-t)} \]
  from Eq. (52) on p. 485.

\textsuperscript{a}Vasicek (1977).
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[ P(t, T) = A(t, T) e^{-B(t, T) r(t)}, \tag{93} \]

where

\[ A(t, T) = \begin{cases} \exp \left[ \frac{(B(t, T) - T + t)(\beta^2 \mu - \sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t, T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\ \exp \left[ \frac{\sigma^2 (T - t)^3}{6} \right] & \text{if } \beta = 0. \end{cases} \]

and

\[ B(t, T) = \begin{cases} \frac{1 - e^{-\beta(T - t)}}{\beta} & \text{if } \beta \neq 0, \\ T - t & \text{if } \beta = 0. \end{cases} \]
The Vasicek Model (concluded)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The spot rate volatility structure is the curve 
  \[ (\partial r(t, T)/\partial r) \sigma = \sigma B(t, T)/(T - t). \]
- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Indeed, higher $\beta$ leads to greater attenuation of volatility with maturity.
The Vasicek Model: Options on Zeros\textsuperscript{a}

- Consider a European call with strike price $X$ expiring at time $T$ on a zero-coupon bond with par value $1$ and maturing at time $s > T$.

- Its price is given by

\[ P(t, s) N(x) - XP(t, T) N(x - \sigma_v). \]

\textsuperscript{a}Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

• Above

\[ x \equiv \frac{1}{\sigma_v} \ln \left( \frac{P(t, s)}{P(t, T) X} \right) + \frac{\sigma_v}{2}, \]

\[ \sigma_v \equiv v(t, T) B(T, s), \]

\[ v(t, T)^2 \equiv \begin{cases} 
\frac{\sigma^2[1-e^{-2\beta(T-t)}]}{2\beta}, & \text{if } \beta \neq 0 \\
\sigma^2(T-t), & \text{if } \beta = 0 
\end{cases}. \]

• By the put-call parity, the price of a European put is

\[ XP(t, T) N(-x + \sigma_v) - P(t, s) N(-x). \]
Binomial Vasicek

- Consider a binomial model for the short rate in the time interval \([0, T]\) divided into \(n\) identical pieces.

- Let \(\Delta t \equiv T/n\) and

\[
p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r) \sqrt{\Delta t}}{2\sigma}.
\]

- The following binomial model converges to the Vasicek model,\(^a\)

\[
r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n.
\]

\(^a\)Nelson and Ramaswamy (1990).
Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$
\text{Prob}[\xi(k) = 1] = \begin{cases} 
p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\
0 & \text{if } p(r(k)) < 0 \\
1 & \text{if } 1 < p(r(k))
\end{cases}.
$$

- Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.

- This is consistent with mean reversion.
Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its constant volatility, $\sigma$.
- For a general process $Y$ with nonconstant volatility, the resulting binomial tree may not combine.
The Cox-Ingersoll-Ross Model

- It is the following square-root short rate model:

\[ dr = \beta (\mu - r) \, dt + \sigma \sqrt{r} \, dW. \] (94)

- The diffusion differs from the Vasicek model by a multiplicative factor \( \sqrt{r} \).

- The parameter \( \beta \) determines the speed of adjustment.

- The short rate can reach zero only if \( 2\beta \mu < \sigma^2 \).

- See text for the bond pricing formula.

---

\( ^a \)Cox, Ingersoll, and Ross (1985).
Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into $n$ periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

- Instead, consider the transformed process

\[ x(r) \equiv 2\sqrt{r}/\sigma. \]

- It follows

\[ dx = m(x) \, dt + dW, \]

where

\[ m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x). \]

- Since this new process has a constant volatility, its associated binomial tree combines.
Binomial CIR (continued)

• Construct the combining tree for $r$ as follows.

• First, construct a tree for $x$.

• Then transform each node of the tree into one for $r$ via the inverse transformation $r = f(x) \equiv x^2 \sigma^2 / 4$ (p. 853).
\[ x \rightarrow x + \sqrt{\Delta t} \rightarrow f(x + \sqrt{\Delta t}) \]
\[ x + \sqrt{\Delta t} \rightarrow x + 2\sqrt{\Delta t} \rightarrow f(x + 2\sqrt{\Delta t}) \]
\[ x \rightarrow x - \sqrt{\Delta t} \rightarrow f(x - \sqrt{\Delta t}) \]
\[ x - \sqrt{\Delta t} \rightarrow x - 2\sqrt{\Delta t} \rightarrow f(x - 2\sqrt{\Delta t}) \]
Binomial CIR (concluded)

• The probability of an up move at each node $r$ is

$$p(r) \equiv \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}. \quad (95)$$

- $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.

- $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

• Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Numerical Examples

• Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

• We shall use \(\Delta t = 0.2 \) (year) for the binomial approximation.

• See p. 856(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (continued)

• Consider the node which is the result of an up move from the root.

• Since the root has $x = 2\sqrt{r(0)/\sigma} = 4$, this particular node’s $x$ value equals $4 + \sqrt{\Delta t} = 4.4472135955$.

• Use the inverse transformation to obtain the short rate $x^2 \times (0.1)^2/4 \approx 0.0494442719102$. 
Numerical Examples (concluded)

• Once the short rates are in place, computing the probabilities is easy.

• Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.

• This phenomenon agrees with mean reversion.

• Convergence is quite good (see text).
A General Method for Constructing Binomial Models\textsuperscript{a}

- We are given a continuous-time process
  \[ dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW. \]

- Make sure the binomial model’s drift and diffusion converge to the above process by setting the probability of an up move to
  \[ \frac{\alpha(y, t) \, \Delta t + y - y_u}{y_u - y_d}. \]

- Here \( y_u \equiv y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

- The displacements are identical, at \( \sigma(y, t) \sqrt{\Delta t} \).

\textsuperscript{a}Nelson and Ramaswamy (1990).
A General Method (continued)

• But the binomial tree may not combine:

\[
\sigma(y, t)\sqrt{\Delta t} - \sigma(y_u, t)\sqrt{\Delta t} \neq -\sigma(y, t)\sqrt{\Delta t} + \sigma(y_d, t)\sqrt{\Delta t}
\]

in general.

• When \( \sigma(y, t) \) is a constant independent of \( y \), equality holds and the tree combines.

• To achieve this, define the transformation

\[
x(y, t) \equiv \int_{y_u}^{y} \sigma(z, t)^{-1} dz.
\]

• Then \( x \) follows \( dx = m(y, t) dt + dW \) for some \( m(y, t) \) (see text).
A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for $x$ combines.

- The probability of an up move remains

$$
\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},
$$

where $y(x, t)$ is the inverse transformation of $x(y, t)$ from $x$ back to $y$.

- Note that $y_u(x, t) \equiv y(x + \sqrt{\Delta t}, t + \Delta t)$ and $y_d(x, t) \equiv y(x - \sqrt{\Delta t}, t + \Delta t)$. 
A General Method (concluded)

• The transformation is
\[ \int_{r}^{r} (\sigma \sqrt{z})^{-1} \, dz = 2\sqrt{r}/\sigma \]

for the CIR model.

• The transformation is
\[ \int_{S}^{S} (\sigma z)^{-1} \, dz = (1/\sigma) \ln S \]

for the Black-Scholes model.

• The familiar binomial option pricing model in fact discretizes \( \ln S \) not \( S \).
Finis