Gamma

- The finite-difference formula for gamma is
  \[ e^{-r\tau} E \left[ \frac{P(S + \epsilon) - 2 \times P(S) + P(S - \epsilon)}{\epsilon^2} \right]. \]

- For a correlation option with multiple underlying assets, the finite-difference formula for the cross gammas \( \frac{\partial^2 P(S_1, S_2, \ldots)}{(\partial S_1 \partial S_2)} \) is:
  \[ e^{-r\tau} E \left[ \frac{P(S_1 + \epsilon_1, S_2 + \epsilon_2) - P(S_1 - \epsilon_1, S_2 + \epsilon_2)}{4\epsilon_1\epsilon_2} \right. \]
  \[ - \left. \frac{P(S_1 + \epsilon_1, S_2 - \epsilon_2) + P(S_1 - \epsilon_1, S_2 - \epsilon_2)}{4\epsilon_1\epsilon_2} \right]. \]
Gamma (concluded)

- Choosing an $\epsilon$ of the right magnitude can be challenging.
  - If $\epsilon$ is too large, inaccurate Greeks result.
  - If $\epsilon$ is too small, unstable Greeks result.
Biases in Pricing Continuously Monitored Options with Monte Carlo

• We are asked to price a continuously monitored up-and-out call with barrier $H$.

• The Monte Carlo method samples the stock price at $n$ discrete time points $t_1, t_2, \ldots, t_n$.

• A sample path $S(t_0), S(t_1), \ldots, S(t_n)$ is produced.
  - Here, $t_0 = 0$ is the current time, and $t_n = T$ is the expiration time of the option.

• If all of the sampled prices are below the barrier, this sample path pays $\max(S(t_n) - X, 0)$.
Biases in Pricing Continuously Monitored Options with Monte Carlo (continued)

- Repeating these steps and averaging the payoffs yield a Monte Carlo estimate.

- This estimate is biased.
  - Suppose none of the sampled prices on a sample path equals or exceeds the barrier $H$.
  - It remains possible for the continuous sample path that passes through them to hit the barrier between sampled time points (see plot on next page).
Biases in Pricing Continuously Monitored Options with Monte Carlo (concluded)

- The bias can certainly be lowered by increasing the number of observations along the sample path.
- However, even daily sampling may not suffice.
- The computational cost also rises as a result.
Brownian Bridge Approach to Pricing Barrier Options

- We desire an unbiased estimate efficiently.
- So the above-mentioned payoff should be multiplied by the probability $p$ that a continuous sample path does not hit the barrier conditional on the sampled prices.
- This methodology is called the Brownian bridge approach.
- Formally, we have

$$p \equiv \text{Prob}[S(t) < H, 0 \leq t \leq T | S(t_0), S(t_1), \ldots, S(t_n)].$$
Brownian Bridge Approach to Pricing Barrier Options (continued)

• As a barrier is hit over a time interval if and only if the maximum stock price over that period is at least $H$,

$$p = \text{Prob} \left[ \max_{0 \leq t \leq T} S(t) < H \mid S(t_0), S(t_1), \ldots, S(t_n) \right].$$

• Luckily, the conditional distribution of the maximum over a time interval given the beginning and ending stock prices is known.
Brownian Bridge Approach to Pricing Barrier Options
(continued)

Lemma 19 Assume $S$ follows $dS/S = \mu \, dt + \sigma \, dW$ and define

$$\zeta(x) \equiv \exp \left[ -\frac{2 \ln(x/S(t)) \ln(x/S(t + \Delta t))}{\sigma^2 \Delta t} \right].$$

(1) If $H > \max(S(t), S(t + \Delta t))$, then

$$\text{Prob} \left[ \max_{t \leq u \leq t + \Delta t} S(u) < H \mid S(t), S(t + \Delta t) \right] = 1 - \zeta(H).$$

(2) If $h < \min(S(t), S(t + \Delta t))$, then

$$\text{Prob} \left[ \min_{t \leq u \leq t + \Delta t} S(u) > h \mid S(t), S(t + \Delta t) \right] = 1 - \zeta(h).$$
Brownian Bridge Approach to Pricing Barrier Options (continued)

- Lemma 19 gives the probability that the barrier is not hit in a time interval, given the starting and ending stock prices.

- For our up-and-out call, choose \( n = 1 \).

- As a result,

\[
p = \begin{cases} 
1 - \exp \left[ -\frac{2 \ln(H/S(0)) \ln(H/S(T))}{\sigma^2 T} \right], & \text{if } H > \max(S(0), S(T)) \\
0, & \text{otherwise.}
\end{cases}
\]
Brownian Bridge Approach to Pricing Barrier Options (continued)

1: \( C := 0; \)
2: \textbf{for} \( i = 1, 2, 3, \ldots, m \) \textbf{do}
3: \hspace{1em} \( P := S \times e^{(r-q-\sigma^2/2)T+\sigma\sqrt{T} \xi();} \)
4: \hspace{1em} \textbf{if} \ (S < H \text{ and } P < H) \text{ or } (S > H \text{ and } P > H) \textbf{then}
5: \hspace{2em} C := C + \max(P-X, 0) \times \left\{ 1 - \exp \left[ -\frac{2 \ln(H/S) \times \ln(H/P)}{\sigma^2 T} \right] \right\};
6: \hspace{1em} \textbf{end if}
7: \textbf{end for}
8: \textbf{return} Ce^{-rT}/m;
Brownian Bridge Approach to Pricing Barrier Options (concluded)

• The idea can be generalized.

• For example, we can handle more complex barrier options.

• Consider an up-and-out call with barrier $H_i$ for the time interval $(t_i, t_{i+1}]$, $0 \leq i < n$.

• This option thus contains $n$ barriers.

• It is a simple matter of multiplying the probabilities for the $n$ time intervals properly to obtain the desired probability adjustment term.
Variance Reduction: Antithetic Variates

• We are interested in estimating $E[g(X_1, X_2, \ldots, X_n)]$, where $X_1, X_2, \ldots, X_n$ are independent.

• Let $Y_1$ and $Y_2$ be random variables with the same distribution as $g(X_1, X_2, \ldots, X_n)$.

• Then

\[
\text{Var} \left[ \frac{Y_1 + Y_2}{2} \right] = \frac{\text{Var}[Y_1]}{2} + \frac{\text{Cov}[Y_1, Y_2]}{2}.
\]

– $\text{Var}[Y_1]/2$ is the variance of the Monte Carlo method with two (independent) replications.

• The variance $\text{Var}[(Y_1 + Y_2)/2]$ is smaller than $\text{Var}[Y_1]/2$ when $Y_1$ and $Y_2$ are negatively correlated.
Variance Reduction: Antithetic Variates (continued)

- For each simulated sample path $X$, a second one is obtained by reusing the random numbers on which the first path is based.
- This yields a second sample path $Y$.
- Two estimates are then obtained: One based on $X$ and the other on $Y$.
- If $N$ independent sample paths are generated, the antithetic-variates estimator averages over $2N$ estimates.
Variance Reduction: Antithetic Variates (continued)

- Consider process \( dX = a_t \, dt + b_t \sqrt{dt} \, \xi \).

- Let \( g \) be a function of \( n \) samples \( X_1, X_2, \ldots, X_n \) on the sample path.

- We are interested in \( E[g(X_1, X_2, \ldots, X_n)] \).

- Suppose one simulation run has realizations \( \xi_1, \xi_2, \ldots, \xi_n \) for the normally distributed fluctuation term \( \xi \).

- This generates samples \( x_1, x_2, \ldots, x_n \).

- The estimate is then \( g(\mathbf{x}) \), where \( \mathbf{x} \equiv (x_1, x_2 \ldots, x_n) \).
Variance Reduction: Antithetic Variates (concluded)

- We do not sample \( n \) more numbers from \( \xi \) for the second estimate.
- The antithetic-variates method computes \( g(x') \) from the sample path \( x' \equiv (x'_1, x'_2 \ldots, x'_n) \) generated by \(-\xi_1, -\xi_2, \ldots, -\xi_n\).
- We then output \( (g(x) + g(x'))/2 \).
- Repeat the above steps for as many times as required by accuracy.
Variance Reduction: Conditioning

• We are interested in estimating $E[X]$.

• Suppose here is a random variable $Z$ such that $E[X | Z = z]$ can be efficiently and precisely computed.

• $E[X] = E[E[X | Z]]$ by the law of iterated conditional expectations.

• Hence the random variable $E[X | Z]$ is also an unbiased estimator of $E[X]$. 
Variance Reduction: Conditioning (concluded)

• As $\text{Var}[E[X \mid Z]] \leq \text{Var}[X]$, $E[X \mid Z]$ has a smaller variance than observing $X$ directly.

• First obtain a random observation $z$ on $Z$.

• Then calculate $E[X \mid Z = z]$ as our estimate.
  – There is no need to resort to simulation in computing $E[X \mid Z = z]$.

• The procedure can be repeated a few times to reduce the variance.
Control Variates

- Use the analytic solution of a similar yet simpler problem to improve the solution.

- Suppose we want to estimate $E[X]$ and there exists a random variable $Y$ with a known mean $\mu \equiv E[Y]$.

- Then $W \equiv X + \beta(Y - \mu)$ can serve as a “controlled” estimator of $E[X]$ for any constant $\beta$.
  - $\beta$ can scale the deviation $Y - \mu$ to arrive at an adjustment for $X$.
  - However $\beta$ is chosen, $W$ remains an unbiased estimator of $E[X]$ as

$$E[W] = E[X] + \beta E[Y - \mu] = E[X].$$
Control Variates (continued)

• Note that

\[ \text{Var}[W] = \text{Var}[X] + \beta^2 \text{Var}[Y] + 2\beta \text{Cov}[X,Y], \]  

(64)

• Hence \( W \) is less variable than \( X \) if and only if

\[ \beta^2 \text{Var}[Y] + 2\beta \text{Cov}[X,Y] < 0. \]  

(65)

• The success of the scheme clearly depends on both \( \beta \) and the choice of \( Y \).
Control Variates (concluded)

• For example, arithmetic average-rate options can be priced by choosing $Y$ to be the otherwise identical geometric average-rate option’s price and $\beta = -1$.

• This approach is much more effective than the antithetic-variates method.
Choice of $Y$

- In general, the choice of $Y$ is ad hoc, and experiments must be performed to confirm the wisdom of the choice.

- Try to match calls with calls and puts with puts.$^a$

- On many occasions, $Y$ is a discretized version of the derivative that gives $\mu$.
  - Discretely monitored geometric average-rate option vs. the continuously monitored geometric average-rate option given by formulas (29) on p. 327.

- For some choices, the discrepancy can be significant, such as the lookback option.$^b$

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$^a$Contributed by Ms. Teng, Huei-Wen (R91723054) on May 25, 2004.

$^b$Contributed by Mr. Tsai, Hwai (R92723049) on May 12, 2004.
Optimal Choice of $\beta$

- Equation (64) on p. 618 is minimized when
  \[
  \beta = -\frac{\text{Cov}[X, Y]}{\text{Var}[Y]},
  \]
  which was called beta earlier in the book.

- For this specific $\beta$,
  \[
  \text{Var}[W] = \text{Var}[X] - \frac{\text{Cov}[X, Y]^2}{\text{Var}[Y]} = \left(1 - \rho^2_{X,Y}\right) \text{Var}[X],
  \]
  where $\rho_{X,Y}$ is the correlation between $X$ and $Y$.

- The stronger $X$ and $Y$ are correlated, the greater the reduction in variance.
Optimal Choice of $\beta$ (continued)

- For example, if this correlation is nearly perfect ($\pm 1$), we could control $X$ almost exactly, eliminating practically all of its variance.

- Typically, neither $\text{Var}[Y]$ nor $\text{Cov}[X, Y]$ is known.

- Therefore, we cannot obtain the maximum reduction in variance.

- We can guess these values and hope that the resulting $W$ does indeed have a smaller variance than $X$.

- A second possibility is to use the simulated data to estimate these quantities.
Optimal Choice of $\beta$ (concluded)

• Observe that $-\beta$ has the same sign as the correlation between $X$ and $Y$.

• Hence, if $X$ and $Y$ are positively correlated, $\beta < 0$, then $X$ is adjusted downward whenever $Y > \mu$ and upward otherwise.

• The opposite is true when $X$ and $Y$ are negatively correlated, in which case $\beta > 0$. 
Problems with the Monte Carlo Method

- The error bound is only probabilistic.
- The probabilistic error bound of $\sqrt{N}$ does not benefit from regularity of the integrand function.
- The requirement that the points be independent random samples are wasteful because of clustering.
- In reality, pseudorandom numbers generated by completely deterministic means are used.
- Monte Carlo simulation exhibits a great sensitivity on the seed of the pseudorandom-number generator.