American Call Pricing in One Period

• Have to consider immediate exercise.

• \( C = \max(hS + B, S - X) \).
  
  – When \( hS + B \geq S - X \), the call should not be exercised immediately.
  
  – When \( hS + B < S - X \), the option should be exercised immediately.

• For non-dividend-paying stocks, early exercise is not optimal by Theorem 3 (p. 186).

• So \( C = hS + B \).
Put Pricing in One Period

• Puts can be similarly priced.

• The delta for the put is \( (P_u - P_d)/(Su - Sd) \leq 0 \), where

\[
\begin{align*}
P_u &= \max(0, X - Su), \\
P_d &= \max(0, X - Sd).
\end{align*}
\]

• Let \( B = \frac{uP_d - dP_u}{(u-d)R} \).

• The European put is worth \( hS + B \).

• The American put is worth \( \max(hS + B, X - S) \).
  
  – Early exercise is always possible with American puts.
Risk

• Surprisingly, the option value is independent of $q$.
• Hence it is independent of the expected gross return of the stock, $qS_u + (1 - q) S_d$.
• It therefore does not directly depend on investors’ risk preferences.
• The option value depends on the sizes of price changes, $u$ and $d$, which the investors must agree upon.
• Note that the possible stock prices are the same whether under $q$ or $p$. 
Pseudo Probability

• After substitution and rearrangement,

\[ hS + B = \frac{\left( \frac{R-d}{u-d} \right) C_u + \left( \frac{u-R}{u-d} \right) C_d}{R}. \]

• Rewrite it as

\[ hS + B = \frac{pC_u + (1-p) C_d}{R}, \]

where

\[ p \equiv \frac{R-d}{u-d}. \]

• As \( 0 < p < 1 \), it may be interpreted as a probability.
Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under $p$ as $pSu + (1 - p) Sd = RS$.
- Risk-neutral investors care only about expected returns.
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, $p$ is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate in a risk-neutral economy.
Binomial Distribution

• Denote the binomial distribution with parameters $n$ and $p$ by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1 - p)^{n-j} = \frac{n!}{j!(n-j)!} p^j (1 - p)^{n-j}.$$  

- $n! = n \times (n - 1) \cdots 2 \times 1$ with the convention $0! = 1$.

• Suppose you toss a coin $n$ times with $p$ being the probability of getting heads.

• Then $b(j; n, p)$ is the probability of getting $j$ heads.
Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.

- Under the binominal model, the stock can take on three possible prices at time two: $S_{uu}$, $S_{ud}$, and $S_{dd}$.
  - There are 4 paths.
  - But the tree combines.

- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

• Let $C_{uu}$ be the call’s value at time two if the stock price is $S_{uu}$.

• Thus,

$$C_{uu} = \max(0, S_{uu} - X).$$

• $C_{ud}$ and $C_{dd}$ can be calculated analogously,

$$C_{ud} = \max(0, S_{ud} - X),$$

$$C_{dd} = \max(0, S_{dd} - X).$$
\[ C_{uu} = \max(0, S_{uu} - X) \]

\[ C_{ud} = \max(0, S_{ud} - X) \]

\[ C_{dd} = \max(0, S_{dd} - X) \]
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

• The call values at time one can be obtained by applying the same logic:

\[
C_u = \frac{pC_{uu} + (1-p)C_{ud}}{R}, \\
C_d = \frac{pC_{ud} + (1-p)C_{dd}}{R}.
\]  

(21)

• Deltas can be derived from Eq. (19) on p. 200.

• For example, the delta at \( C_u \) is

\[
\frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}}.
\]
Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- An equivalent portfolio of \( h \) shares of stock and \( B \) riskless bonds can be set up for the call that costs \( C_u \) (\( C_d \), resp.) if the stock price goes to \( S_u \) (\( S_d \), resp.).
- The values of \( h \) and \( B \) can be derived from Eqs. (19)–(20) on p. 200.
- Or, we can just compute

\[
\frac{pC_u + (1 - p) C_d}{R}
\]

as the price.
Early Exercise

• Since the call will not be exercised at time one even if it is American, \( C_u \geq Su - X \) and \( C_d \geq Sd - X \).

• Therefore,

\[
\begin{align*}
hS + B &= \frac{pC_u + (1 - p)C_d}{R} \\
&\geq \left[ pu + (1 - p) d \right] S - X \\
&= S - \frac{X}{R} > S - X.
\end{align*}
\]

• The call again will not be exercised at present.

• So

\[
C = hS + B = \frac{pC_u + (1 - p)C_d}{R}.
\]
Backward Induction of Zermelo (1871–1953)

• The above expression calculates $C$ from the two successor nodes $C_u$ and $C_d$ and none beyond.

• The same computation happens at $C_u$ and $C_d$, too, as demonstrated in Eq. (21) on p. 211.

• This recursive procedure is called backward induction.

• Now, $C$ equals

\[
\begin{align*}
[p^2 C_{uu} + 2p(1-p) C_{ud} + (1-p)^2 C_{dd}] &\left(\frac{1}{R^2}\right) \\
= \left[p^2 \max(0, S_{u^2} - X) + 2p(1-p) \max(0, S_{ud} - X) \right. \\
&\left. + (1-p)^2 \max(0, S_{d^2} - X) \right]/R^2.
\end{align*}
\]
$$S_0 u^2$$

$$S_0 u$$

$$S_0 ud$$

$$S_0 d$$

$$S_0 d^2$$

$$1$$

$$p$$

$$p^2$$

$$2p(1 - p)$$

$$1 - p$$

$$(1 - p)^2$$
Backward Induction (concluded)

• In the $n$-period case,

\[
C = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max \left(0, S u^j d^{n-j} - X\right) \frac{R^n}{R^n}.
\]

– The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

• The value of a European put is

\[
P = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max \left(0, X - S u^j d^{n-j}\right) \frac{R^n}{R^n}.
\]
Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.

- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^\pi [\mathcal{D}].$$

- $E^\pi$ means the expectation is taken under the risk-neutral probability.

- The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.
Self-Financing

• Delta changes over time.

• The maintenance of an equivalent portfolio is dynamic.

• The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.

• The portfolio’s value at the end of the current period is precisely the amount needed to set up the next portfolio.

• The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.
  – Changes in value are due entirely to capital gains.
The Binomial Option Pricing Formula

- Let \( a \) be the minimum number of upward price moves for the call to finish in the money.

- So \( a \) is the smallest nonnegative integer such that

\[
Su^a d^{n-a} \geq X,
\]

or

\[
a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.
\]
The Binomial Option Pricing Formula (concluded)

Hence,

\[
C = \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} (S w^j d^{n-j} - X)}{R^n} \tag{22}
\]

\[
= S \sum_{j=a}^{n} \binom{n}{j} \left( \frac{pu}{R^n} \right)^j \left[ \frac{(1 - p) d}{R^n} \right]^{n-j} - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j}
\]

\[
= S \sum_{j=a}^{n} b(j; n, pue^{-\hat{r}}) - X e^{-\hat{r}n} \sum_{j=a}^{n} b(j; n, p).
\]
Numerical Examples

- A non-dividend-paying stock is selling for $160.
- \( u = 1.5 \) and \( d = 0.5 \).
- \( r = 18.232\% \) per period \((e^{0.18232} = 1.2)\).
- Consider a European call on this stock with \( X = 150 \) and \( n = 3 \).
- The call value is $85.069 by backward induction.
- Also the PV of the expected payoff at expiration,

\[
\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.
\]
Binomial process for the stock price (probabilities in parentheses)

- 160
  - 80 (0.3)
    - 40 (0.09)
      - 20 (0.027)
    - 60 (0.189)
  - 120 (0.42)
    - 240 (0.7)
      - 360 (0.49)
      - 540 (0.343)
  - 360 (0.49)
    - 180 (0.441)

Binomial process for the call price (hedge ratios in parentheses)

- 160
  - 80 (0.3)
    - 40 (0.09)
      - 20 (0.027)
    - 60 (0.189)
  - 120 (0.42)
    - 240 (0.7)
      - 360 (0.49)
      - 540 (0.343)
  - 360 (0.49)
    - 180 (0.441)
      - 235 (1.0)
      - 30 (0.90625)
      - 17.5 (0.25)
      - 0 (0.0)
Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for $90 instead.
- Sell the call for $90 and invest $85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

\[ 90 - 85.069 = 4.931 \text{ dollars}, \]

is the arbitrage profit as we will see.
Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to $240.
- The new delta is 0.90625.
- Buy $0.90625 - 0.82031 = 0.08594$ more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.
Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to $120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to $76.04232 \times 1.2 - 78.75 = 12.5$ dollars.
Numerical Examples (continued)

Time 3 (the case of rising price):

• The stock price moves to $180.

• The call we wrote finishes in the money.

• For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.

• Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.

• It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.
Numerical Examples (concluded)

Time 3 (the case of declining price):

• The stock price moves to $60.
• The call we wrote is worthless.
• Sell the 0.25 shares of stock for a total of $0.25 \times 60 = 15$ dollars.
• Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.
Binomial Tree Algorithms for European Options

• The BOPM implies the binomial tree algorithm that applies backward induction.

• The total running time is $O(n^2)$.

• The memory requirement is $O(n^2)$.
  – Can be further reduced to $O(n)$ by reusing space

• To price European puts, simply replace the payoff.
Further Improvement for Calls
Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

- Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p)^j} b(j - 1; n, p).$$
Optimal Algorithm (continued)

• The following program computes $b(j; n, p)$ in $b[j]$:

1: $b[a] := \binom{n}{a} p^a (1 - p)^{n-a}$;

2: for $j = a + 1, a + 2, \ldots, n$ do

3: $b[j] := b[j - 1] \times p \times (n - j + 1)/((1 - p) \times j)$;

4: end for

• It runs in $O(n)$ steps.
Optimal Algorithm (concluded)

• With the $b(j; n, p)$ available, the risk-neutral valuation formula (22) on p. 220 is trivial to compute.

• We only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.

• This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.

• The above technique cannot be applied to American options because of early exercise.

• So binomial tree algorithms for American options usually run in $O(n^2)$ time.
On the Bushy Tree
Toward the Black-Scholes Formula

• The binomial model seems to suffer from two unrealistic assumptions.
  – The stock price takes on only two values in a period.
  – Trading occurs at discrete points in time.

• As the number of periods increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.

• Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.

• We now skim through the proof.
Toward the Black-Scholes Formula (continued)

• Let $\tau$ denote the time to expiration of the option measured in years.

• Let $r$ be the continuously compounded annual rate.

• With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.

• Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n$ goes to infinity.

• First, $\hat{r} = r\tau/n$.
  
  – The period gross return $R = e^{\hat{r}}$. 
Toward the Black-Scholes Formula (continued)

• Use

$$\hat{\mu} \equiv \frac{1}{n} E \left[ \ln \frac{S_\tau}{S} \right] \quad \text{and} \quad \hat{\sigma}^2 \equiv \frac{1}{n} \text{Var} \left[ \ln \frac{S_\tau}{S} \right]$$

to denote, resp., the expected value and variance of the period continuously compounded rate of return.

• Under the BOPM, it is not hard to show that

$$\hat{\mu} = q \ln(u/d) + \ln d,$$
$$\hat{\sigma}^2 = q(1 - q) \ln^2(u/d).$$
Toward the Black-Scholes Formula (continued)

- Assume the stock’s true continuously compounded rate of return over $\tau$ years has mean $\mu \tau$ and variance $\sigma^2 \tau$.
  - Call $\sigma$ the stock’s (annualized) volatility.

- The BOPM converges to the distribution only if
  \[
  n \hat{\mu} = n(q \ln(u/d) + \ln d) \to \mu \tau, \\
  n \hat{\sigma}^2 = nq(1 - q) \ln^2(u/d) \to \sigma^2 \tau.
  \]

- Impose $ud = 1$ to make nodes at the same horizontal level of the tree have identical price (review p. 230).
  - Other choices are possible (see text).
Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

\[ u = e^{\sigma \sqrt{\frac{\tau}{n}}}, \quad d = e^{-\sigma \sqrt{\frac{\tau}{n}}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\frac{\tau}{n}}. \]  

(23)

- With Eqs. (23),

\[ n\hat{\mu} = \mu \tau, \]

\[ n\hat{\sigma}^2 = \left[ 1 - \left( \frac{\mu}{\sigma} \right)^2 \frac{\tau}{n} \right] \sigma^2 \tau \rightarrow \sigma^2 \tau. \]
Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities \( u > R > d \) may not hold under Eqs. (23) on p. 239.
  - If this happens, the risk-neutral probability may lie outside \([0, 1]\).
- The problem disappears when \( n \) satisfies
  \[
e^{\sigma\sqrt{\tau/n}} > e^{r\tau/n},
\]
in other words, when \( n > \frac{r^2\tau}{\sigma^2} \) (check it).
  - So it goes away if \( n \) is large enough.
  - Other solutions will be presented later.
Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau / S)$?

- The central limit theorem says $\ln(S_\tau / S)$ converges to the normal distribution with mean $\mu \tau$ and variance $\sigma^2 \tau$.

- So $\ln S_\tau$ approaches the normal distribution with mean $\mu \tau + \ln S$ and variance $\sigma^2 \tau$.

- $S_\tau$ has a lognormal distribution in the limit.
Toward the Black-Scholes Formula (continued)

**Lemma 7** The continuously compounded rate of return
\[ \ln\left(\frac{S_\tau}{S}\right) \]
approaches the normal distribution with mean
\[ (r - \sigma^2/2) \tau \]
and variance \( \sigma^2 \tau \) in a risk-neutral economy.

- Let \( q \) equal the risk-neutral probability
  \[ p \equiv \frac{(e^{r\tau/n} - d)}{(u - d)}. \]
- Let \( n \to \infty \).
Toward the Black-Scholes Formula (continued)

• By Lemma 7 (p. 242) and Eq. (17) on p. 147, the expected stock price at expiration in a risk-neutral economy is $S e^{r \tau}$.

• The stock’s expected annual rate of return\(^{a}\) is thus the riskless rate \(r\).

\(^{a}\)In the sense of \((1/\tau) \ln E[S_{\tau}/S]\) not \((1/\tau)E[\ln(S_{\tau}/S)]\).
Toward the Black-Scholes Formula (concluded)

Theorem 8 (The Black-Scholes Formula)

\[ C = SN(x) - X e^{-r\tau} N(x - \sigma \sqrt{\tau}), \]
\[ P = X e^{-r\tau} N(-x + \sigma \sqrt{\tau}) - SN(-x), \]

where

\[ x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}. \]
**BOPM and Black-Scholes Model**

- The Black-Scholes formula needs five parameters: $S$, $X$, $\sigma$, $\tau$, and $r$.

- Binomial tree algorithms take six inputs: $S$, $X$, $u$, $d$, $\hat{r}$, and $n$.

- The connections are

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad \hat{r} = r\tau/n.$$  

- The binomial tree algorithms converge reasonably fast.

- Oscillations can be dealt with by the judicious choices of $u$ and $d$ (see text).
\( S = 100, \ X = 100 \) (left), and \( X = 95 \) (right).
Implied Volatility

• Volatility is the sole parameter not directly observable.

• The Black-Scholes formula can be used to compute the market’s opinion of the volatility.
  – Solve for $\sigma$ given the option price, $S$, $X$, $\tau$, and $r$ with numerical methods.
  – How about American options?

• This volatility is called the implied volatility.

• Implied volatility is often preferred to historical volatility in practice.\textsuperscript{a}

\textsuperscript{a}It is like driving a car with your eyes on the rearview mirror?