Price Behavior (2)

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.
<table>
<thead>
<tr>
<th>Yield (%)</th>
<th>Price (% of par)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>113.37</td>
</tr>
<tr>
<td>8.0</td>
<td>108.65</td>
</tr>
<tr>
<td>8.5</td>
<td>104.19</td>
</tr>
<tr>
<td>9.0</td>
<td>100.00</td>
</tr>
<tr>
<td>9.5</td>
<td>96.04</td>
</tr>
<tr>
<td>10.0</td>
<td>92.31</td>
</tr>
<tr>
<td>10.5</td>
<td>88.79</td>
</tr>
</tbody>
</table>
Terminology

• Bonds selling at par are called par bonds.
• Bonds selling at a premium are called premium bonds.
• Bonds selling at a discount are called discount bonds.
Price Behavior (3): Convexity
Day Count Conventions: Actual/Actual

- The first “actual” refers to the actual number of days in a month.

- The second refers to the actual number of days in a coupon period.

- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.
Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is
  \[360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1)\]
Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.

- Let

\[ \omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \] (5)

- The price is now calculated by

\[ PV = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}. \] (6)
Accrued Interest

• The buyer pays the quoted price plus the accrued interest — the invoice price:

\[ C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega). \]

• The yield to maturity is the \( r \) satisfying Eq. (6) when \( P \) is the invoice price.

• The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
$C(1 - \omega)$

coupon payment date

$(1 - \omega)\%$

$\omega\%$

coupon payment date
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
Example ("30/360") (concluded)

• The accrued interest is \((10/2) \times \frac{180 - 60}{180} = 3.3333\) per $100 of par value.

• The yield to maturity is 3%.

• This can be verified by Eq. (6) on p. 67 with
  - \(\omega = 60/180\),
  - \(m = 2\),
  - \(C = 5\),
  - \(PV = 111.2891 + 3.3333\),
  - \(r = 0.03\).
Price Behavior (2) Revisited

• Before: A bond selling at par if the yield to maturity equals the coupon rate.

• But it assumed that the settlement date is on a coupon payment date.

• Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.

• Then its yield to maturity is less than the coupon rate.
  – The short reason: Exponential growth is replaced by linear growth, hence “overpaying” the coupon.
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

• Volatility measures how bond prices respond to interest rate changes.

• It is key to the risk management of interest-rate-sensitive securities.

• Assume level-coupon bonds throughout.
Price Volatility (concluded)

• What is the sensitivity of the percentage price change to changes in interest rates?

• Define price volatility by

\[ - \frac{\partial P}{\partial y} \cdot \frac{1}{P}. \]
Price Volatility of Bonds

• The price volatility of a coupon bond is

\[
- \frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)}.
\]

– $F$ is the par value.
– $C$ is the coupon payment per period.

• For bonds without embedded options,

\[
- \frac{\partial P}{\partial y} > 0.
\]
Macaulay Duration

• The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.

• The weights are the cash flows’ PVs divided by the asset’s price.

• Formally,

\[
MD \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1 + y)^i}.
\]

• The Macaulay duration, in periods, is equal to

\[
MD = -(1 + y) \frac{\partial P}{\partial y} \frac{1}{P}.
\]  (7)
MD of Bonds

- The MD of a coupon bond is

$$\text{MD} = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1 + y)^i} + \frac{nF}{(1 + y)^n} \right]. \quad (8)$$

- It can be simplified to

$$\text{MD} = \frac{c(1 + y) \left[ (1 + y)^n - 1 \right] + ny(y - c)}{cy \left[ (1 + y)^n - 1 \right] + y^2},$$

where $c$ is the period coupon rate.

- The MD of a zero-coupon bond equals its term to maturity $n$.

- The MD of a coupon bond is less than its maturity.
Finesse

• Equations (7) on p. 78 and (8) on p. 79 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the MD may actually decrease.
How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But you use it that way at your peril.
- The MD should be seen mainly as measuring price volatility.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.
Conversion

• For the MD to be year-based, modify Eq. (8) on p. 79 to

\[
\frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \frac{C}{(1 + \frac{y}{k})^i} + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
\]

where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

• Equation (7) on p. 78 also becomes

\[
MD = - \left( 1 + \frac{y}{k} \right) \frac{\partial P}{\partial y} \frac{1}{P}.
\]

• By definition, MD (in years) = \( \frac{MD \text{ (in periods)}}{k} \).
Modified Duration

- Modified duration is defined as

\[
\text{modified duration} \equiv -\frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}.
\]  \hspace{5em} (9)

- By Taylor expansion,

percent price change \(\approx -\) modified duration \(\times\) yield change.
Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

\[-11.54 \times 0.001 = -0.01154 = -1.154\%.
]
Modified Duration of a Portfolio

- The modified duration of a portfolio equals
  \[ \sum_i \omega_i D_i. \]
  
  - \( D_i \) is the modified duration of the \( i \)th asset.
  - \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as
  \[
  \frac{P_- - P_+}{P_0(y_+ - y_-)}.
  \]
  - \(P_-\) is the price if the yield is decreased by \(\Delta y\).
  - \(P_+\) is the price if the yield is increased by \(\Delta y\).
  - \(P_0\) is the initial price, \(y\) is the initial yield.
  - \(\Delta y\) is small.
Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use

\[ \frac{P_0 - P_+}{P_0 \Delta y} \]

- More economical but less accurate.
The Practices

- Duration is usually expressed in percentage terms—call it $D\%$—for quick mental calculation.

- The percentage price change expressed in percentage terms is approximated by

  $$-D\% \times \Delta r$$

  when the yield increases instantaneously by $\Delta r\%$.

  - Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

- In fact, $D\%$ equals modified duration as originally defined (prove it!).
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

\[
\text{modified duration} \times \text{price (\% of par)} = -\frac{\partial P}{\partial y}.
\]

- The approximate dollar price change per $100 of par value is

\[
\text{price change} \approx -\text{dollar duration} \times \text{yield change}.
\]
Convexity

• Convexity is defined as

\[
\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.
\]

• The convexity of a coupon bond is positive (prove it!).

• For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

• Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.
Convexity (concluded)

• Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}$$

when there are $k$ periods per annum.
Use of Convexity

• The approximation $\Delta P/P \approx -\text{duration} \times \text{yield change}$ works for small yield changes.

• To improve upon it for larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$  

• Recall the figure on p. 91.
The Practices

- Convexity is usually expressed in percentage terms—call it $C\%$—for quick mental calculation.

- The percentage price change expressed in percentage terms is approximated by $-D\% \times \Delta r + C\% \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
  
  - Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because
    
    $$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

- In fact, $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

- The effective convexity is defined as
  \[ \frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2}, \]
  - \(P_-\) is the price if the yield is decreased by \(\Delta y\).
  - \(P_+\) is the price if the yield is increased by \(\Delta y\).
  - \(P_0\) is the initial price, \(y\) is the initial yield.
  - \(\Delta y\) is small.

- Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

- Numerically, choosing the right \(\Delta y\) is a delicate matter.
Approximate \( d^2 f(x)^2 / dx^2 \) at \( x = 1 \), Where \( f(x) = x^2 \)

The difference of \( ((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2 \) and 2:
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest. Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

• Concerned with how interest rates change with maturity.

• The set of yields to maturity for bonds forms the term structure.
  – The bonds must be of equal quality.
  – They differ solely in their terms to maturity.

• The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is
  \[
  [1 + S(i)]^{-i}.
  \]
- The one-period spot rate is called the short rate.
- A spot rate curve is a plot of spot rates against maturity.
Problems with the PV Formula

- In the bond price formula,

\[ \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n}, \]

every cash flow is discounted at the same yield \( y \).

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams.

- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.

- But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

- A cash flow \( C_1, C_2, \ldots, C_n \) is equivalent to a package of zero-coupon bonds with the \( i \)th bond paying \( C_i \) dollars at time \( i \).

- So a level-coupon bond has the price

\[
P = \sum_{i=1}^{n} \frac{C_i}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \tag{10}
\]

- This pricing method incorporates information from the term structure.

- Discount each cash flow at the corresponding spot rate.
Discount Factors

• In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

  - Above, $d(i) \equiv [1 + S(i)]^{-i}, i = 1, 2, \ldots, n$, are called discount factors.
  - $d(i)$ is the PV of one dollar $i$ periods from now.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Note that short-term Treasuries are zero-coupon bonds.

- Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.

- Then $S(n)$ can be computed from Eq. (10) on p. 105, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1+S(i)]^i} + \frac{F}{[1+S(n)]^n}.$$

- The running time is $O(n)$ (see text).

- The procedure is called bootstrapping.
Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  - But, any economic justifications?
Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for $P$.
- Were this bond riskless, it would fetch
  \[ P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}. \]
- Since riskiness must be compensated, $P < P^*$.
- Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.
Static Spread

- The static spread is the amount $s$ by which the spot rate curve has to shift in parallel to price the risky bond:

$$P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}.$$ 

- Unlike the yield spread, the static spread incorporates information from the term structure.
Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

• The spot rate curve often has the same shape as the yield curve.
  – If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

• But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.
- Invest $1 for \( j \) periods to end up with \( [1 + S(j)]^j \) dollars at time \( j \).
  - The maturity strategy.
- Invest $1 in bonds for \( i \) periods and at time \( i \) invest the proceeds in bonds for another \( j - i \) periods where \( j > i \).
- Will have \( [1 + S(i)]^i[1 + S(i, j)]^{j-i} \) dollars at time \( j \).
  - \( S(i, j) \): \((j - i)\)-period spot rate \( i \) periods from now.
  - The rollover strategy.
Forward Rates (concluded)

• When $S(i, j)$ equals

$$f(i, j) \equiv \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1,$$  \hspace{1cm} (11)

we will end up with $[1 + S(j)]^j$ dollars again.

• By definition, $f(0, j) = S(j)$.

• $f(i, j)$ is called the (implied) forward rates.
  – More precisely, the $(j - i)$-period forward rate $i$ periods from now.
The diagram illustrates a time line with various intervals labeled as $f(0,1)$, $f(1,2)$, $f(2,3)$, and $f(3,4)$. At time 0, events $S(1)$, $S(2)$, $S(3)$, and $S(4)$ are depicted moving along the timeline.
Forward Rates and Future Spot Rates

• We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  – This is the subject of the term structure theories.

• We merely looked for the future spot rate that, if realized, will equate two investment strategies.

• $f(i, i + 1)$ are instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

\[ f(i, j) > S(j) > \cdots > S(i). \]

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

\[ f(i, j) < S(j) < \cdots < S(i). \]
Forward Rates ≡ Spot Rates ≡ Yield Curve

- The FV of $1 at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive
  \[ [1 + S(n)]^n. \]
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is
  \[ [1 + S(1)] [1 + f(1, 2)] \cdots [1 + f(n - 1, n)]. \]
Forward Rates ≡ Spot Rates ≡ Yield Curves (concluded)

- Since they are identical,

\[ S(n) = \left\{ \left[ 1 + S(1) \right] \left[ 1 + f(1, 2) \right] \right. \]
\[ \cdot \cdots \cdot \left[ 1 + f(n - 1, n) \right] \}^{1/n} - 1. \quad (12) \]

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.

- Other equivalencies can be derived similarly, such as

\[ f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1. \]