

Option Pricing Models

The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value.
- Need a model of probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.
 - Known as the Black-Scholes option pricing model.

If the world of sense does not fit mathematics,
so much the worse for the world of sense.
— Bertrand Russell (1872–1970)

Terms and Approach

- C : call value.
- P : put value.
- X : strike price
- S : stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

Binomial Option Pricing Model (BOPM)

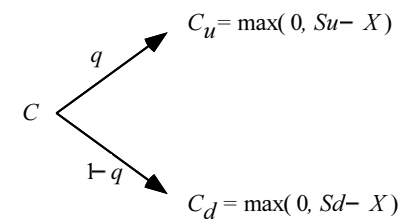
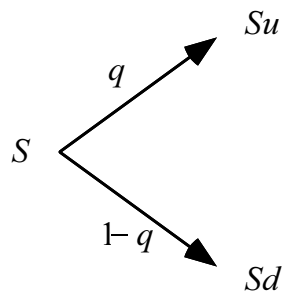
- Time is discrete and measured in periods.
- If the current stock price is S , it can go to Su with probability q and Sd with probability $1 - q$, where $0 < q < 1$ and $d < u$.
 - In fact, $d < R < u$ must hold to rule out arbitrage.
- Six pieces of information suffice to determine the option value based on arbitrage considerations: S , u , d , X , \hat{r} , and the number of periods to expiration.

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the call price at time one if the stock price moves to Su .
- C_d is the call price at time one if the stock price moves to Sd .
- Clearly,

$$C_u = \max(0, Su - X),$$

$$C_d = \max(0, Sd - X).$$



Call on a Non-Dividend-Paying Stock: Single Period (continued)

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
 - This costs $hS + B$.
 - We call h the hedge ratio or delta.
- The value of this portfolio at time one is either $hSu + RB$ or $hSd + RB$.
- Choose h and B such that the portfolio replicates the payoff of the call,

$$hSu + RB = C_u,$$

$$hSd + RB = C_d.$$

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
 - When $hS + B \geq S - X$, the call should not be exercised immediately.
 - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 3 (p. 183).
- So $C = hS + B$.

Call on a Non-Dividend-Paying Stock: Single Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \quad (19)$$

$$B = \frac{uC_d - dC_u}{(u - d)R}. \quad (20)$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio, $C = hS + B$.
- As $uC_d - dC_u < 0$, the equivalent portfolio is a levered long position in stocks.

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$P_u = \max(0, X - Su),$$

$$P_d = \max(0, X - Sd).$$
- Let $B = \frac{uP_d - dP_u}{(u - d)R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.
 - Early exercise is always possible with American puts.

Risk

- Surprisingly, the option value is independent of q .
- Hence it is independent of the expected gross return of the stock, $qSu + (1 - q)Sd$.
- It therefore does not directly depend on investors' risk preferences.
- The option value depends on the sizes of price changes, u and d , which the investors must agree upon.
- Note that the possible stock prices are the same whether under q or p .

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate \hat{r} under p as $pSu + (1 - p)Sd = RS$.
- Risk-neutral investors care only about expected returns.
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate in a risk-neutral economy.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}.$$

- Rewrite it as

$$hS + B = \frac{pC_u + (1-p)C_d}{R},$$

where

$$p \equiv \frac{R-d}{u-d}.$$

- As $0 < p < 1$, it may be interpreted as a probability.

Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}.$$

$$- n! = n \times (n-1) \cdots 2 \times 1 \text{ with the convention } 0! = 1.$$

- Suppose you toss a coin n times with p being the probability of getting heads.
- Then $b(j; n, p)$ is the probability of getting j heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: S_{uu} , S_{ud} , and S_{dd} .
 - Note that the tree combines.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

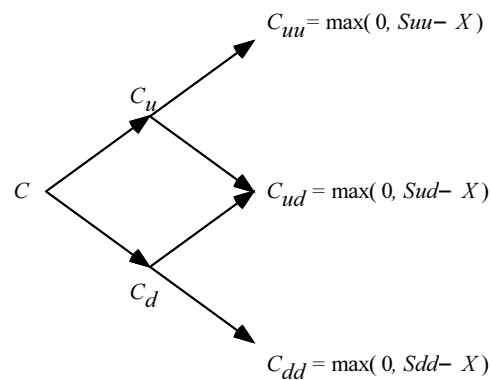
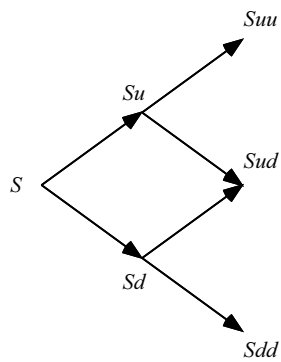
- Let C_{uu} be the call's value at time two if the stock price is S_{uu} .
- Thus,

$$C_{uu} = \max(0, S_{uu} - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$C_{ud} = \max(0, S_{ud} - X),$$

$$C_{dd} = \max(0, S_{dd} - X).$$



Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time one can be obtained by applying the same logic:

$$\begin{aligned} C_u &= \frac{pC_{uu} + (1-p)C_{ud}}{R}, \\ C_d &= \frac{pC_{ud} + (1-p)C_{dd}}{R}. \end{aligned} \quad (21)$$

- Deltas can be derived from Eq. (19) on p. 197.
- For example, the delta at C_u is

$$\frac{C_{uu} - C_{ud}}{Su_u - Sud}.$$

Early Exercise

- Since the call will not be exercised at time one even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)C_d}{R} \geq \frac{[pu + (1-p)d]S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

- The call again will not be exercised at present.
- So

$$C = hS + B = \frac{pC_u + (1-p)C_d}{R}.$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- An equivalent portfolio of h shares of stock and B riskless bonds can be set up for the call that costs C_u (C_d , resp.) if the stock price goes to Su (Sd , resp.).
- The values of h and B can be derived from Eqs. (19)–(20) on p. 197.
- Or, we can just compute

$$\frac{pC_u + (1-p)C_d}{R}$$

as the price.

Backward Induction of Zermelo (1871–1953)

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happens at C_u and C_d , too, as demonstrated in Eq. (21) on p. 208.
- This recursive procedure is called backward induction.
- Now, C equals

$$\begin{aligned} &[p^2C_{uu} + 2p(1-p)C_{ud} + (1-p)^2C_{dd}]/R^2 \\ &= [p^2 \max(0, Su^2 - X) + 2p(1-p) \max(0, Sud - X) \\ &\quad + (1-p)^2 \max(0, Sd^2 - X)]/R^2. \end{aligned}$$

Backward Induction (continued)

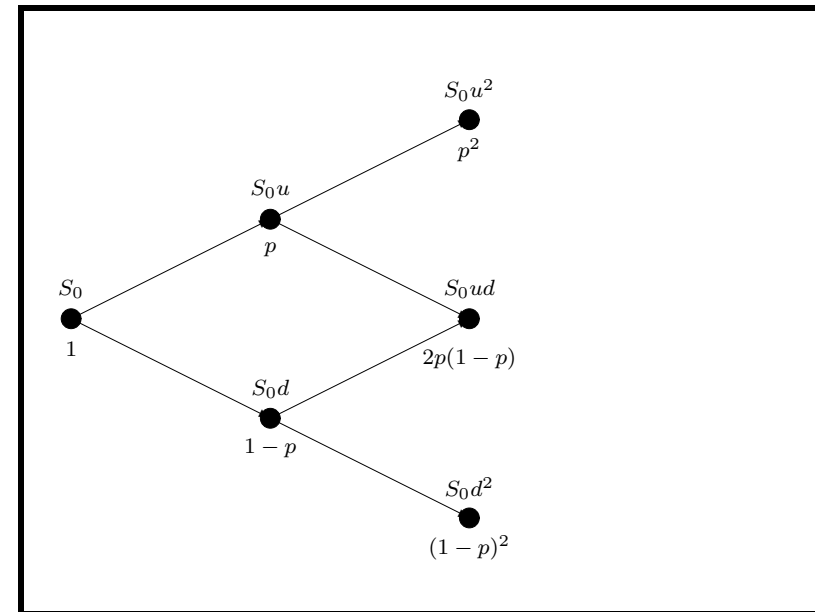
- In the n -period case,

$$C = \frac{\sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \times \max(0, Su^j d^{n-j} - X)}{R^n}.$$

- The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- The value of a European put is

$$P = \frac{\sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \times \max(0, X - Su^j d^{n-j})}{R^n}.$$



Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function \mathcal{D} , its value is

$$e^{-\hat{r}n} E^\pi[\mathcal{D}].$$

- E^π means the expectation is taken under the risk-neutral probability.
- The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio’s value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.
 - Changes in value are due entirely to capital gains.

The Binomial Option Pricing Formula

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer such that

$$Su^a d^{n-a} \geq X,$$

or

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

Numerical Examples

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($e^{0.18232} = 1.2$).
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Also the PV of the expected payoff at expiration,

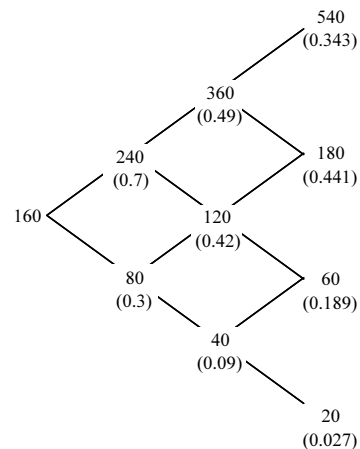
$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$

The Binomial Option Pricing Formula (concluded)

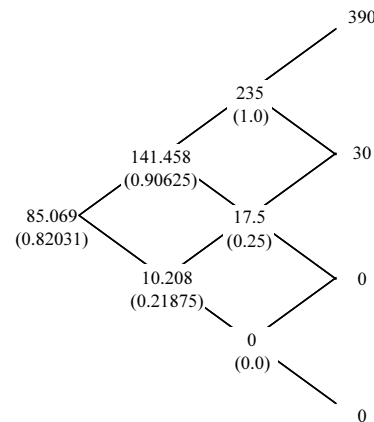
Hence,

$$\begin{aligned} C &= \frac{\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} (Su^j d^{n-j} - X)}{R^n} \\ &= S \sum_{j=a}^n \binom{n}{j} \frac{(pu)^j [(1-p)d]^{n-j}}{R^n} - \frac{X}{R^n} \sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} \\ &= S \sum_{j=a}^n b(j; n, pue^{-\hat{r}}) - Xe^{-\hat{r}n} \sum_{j=a}^n b(j; n, p). \end{aligned} \quad (22)$$

Binomial process for the stock price
(probabilities in parentheses)



Binomial process for the call price
(hedge ratios in parentheses)



Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars,}$$

is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to $76.04232 \times 1.2 - 78.75 = 12.5$ dollars.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy $0.90625 - 0.82031 = 0.08594$ more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

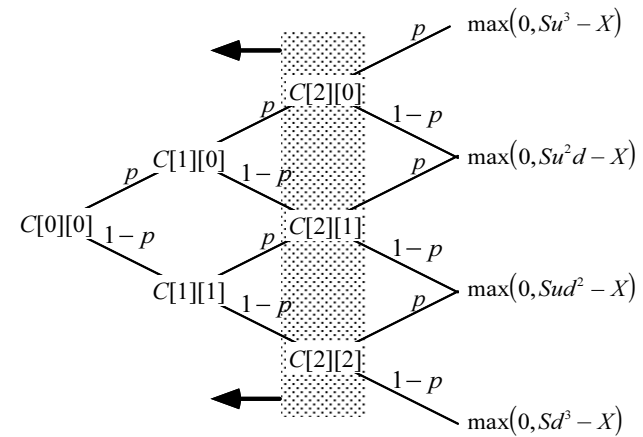
Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

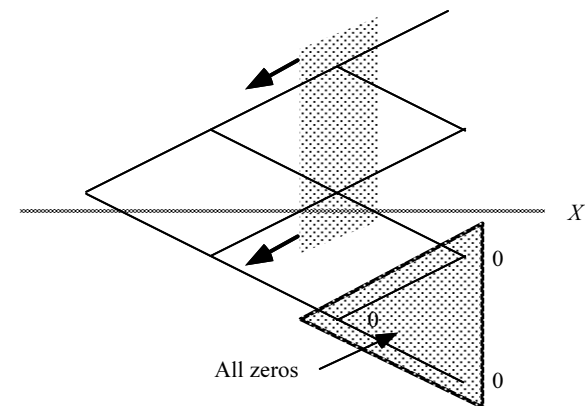
- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of $0.25 \times 60 = 15$ dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.



Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n^2)$.
 - Can be further reduced to $O(n)$ by reusing space
- To price European puts, simply replace the payoff.

Further Improvement for Calls



Optimal Algorithm

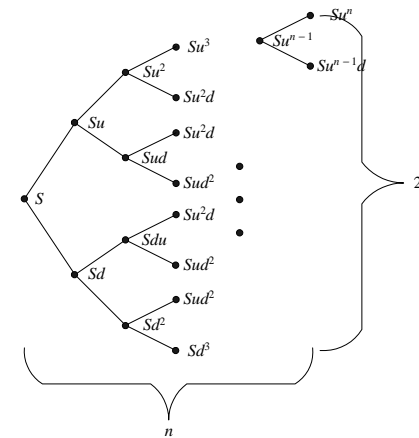
- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

- Note that

$$b(j; n, p) = \frac{p(n-j+1)}{(1-p)j} b(j-1; n, p).$$

- The following program computes $b(j; n, p)$ in $b[j]$,
 - 1: $b[a] := \binom{n}{a} p^a (1-p)^{n-a}$;
 - 2: **for** $j = a+1, a+2, \dots, n$ **do**
 - 3: $b[j] := b[j-1] \times p \times (n-j+1) / ((1-p) \times j)$;
 - 4: **end for**
- It runs in $O(n)$ steps.

On the Bushy Tree



Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (22) on p. 217 is trivial to compute.
- We only need a single variable to store the $b(j; n, p)$ s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique cannot be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.