Conditional Variance Models for Price Volatility

- Although a stationary model (see text for definition) has constant variance, its conditional variance may vary.
- Take for example an AR(1) process $X_t = aX_{t-1} + \epsilon_t$ with $|a| < 1$.
  - Here, $\epsilon_t$ is a stationary, uncorrelated process with zero mean and constant variance $\sigma^2$.
- The conditional variance,
  \[
  \text{Var}[X_t | X_{t-1}, X_{t-2}, \ldots],
  \]
  equals $\sigma^2$, which is smaller than the unconditional variance $\text{Var}[X_t] = \sigma^2/(1-a^2)$.

Conditional Variance Models for Price Volatility (continued)

- Past information thus has no effects on the variance of prediction.
- To address this drawback, consider models for returns $X_t$ consistent with a changing conditional variance:
  \[
  X_t - \mu = V_t U_t.
  \]
  - $U_t$ has zero mean and unit variance for all $t$.
  - $E[X_t] = \mu$ for all $t$.
  - $\text{Var}[X_t | V_t = v_t] = v_t^2$. 
Conditional Variance Models for Price Volatility (continued)

- The process \( \{ V_t^2 \} \) models the conditional variance.
- Suppose \( \{ U_t \} \) and \( \{ V_t \} \) are independent of each other, which means \( \{ U_1, U_2, \ldots, U_n \} \) and \( \{ V_1, V_2, \ldots, V_n \} \) are independent for all \( n \).
- Then \( \{ X_t \} \) is uncorrelated because
  \[
  \text{Cov}[X_t, X_{t+\tau}] = 0 \quad (77)
  \]
  for \( \tau > 0 \) (see text for proof).

- If, furthermore, \( \{ V_t \} \) is stationary, then \( \{ X_t \} \) has constant variance because
  \[
  E[(X_t - \mu)^2]
  = E[V_t^2 U_t^2]
  = E[V_t^2] E[U_t^2]
  = E[V_t^2].
  \]
  This makes \( \{ X_t \} \) stationary.

Conditional Variance Models for Price Volatility (continued)

- In the lognormal model, the conditional variance evolves independently of past returns.
- Suppose we assume that conditional variances are deterministic functions of past returns:
  \[ V_t = f(X_{t-1}, X_{t-2}, \ldots) \]
  for some function \( f \).
- Then \( V_t \) can be computed given the information set of past returns:
  \[ I_{t-1} \equiv \{ X_{t-1}, X_{t-2}, \ldots \} \].

ARCH Models\(^a\)

- An influential model in this direction is the autoregressive conditional heteroskedastic (ARCH) model.
- Assume \( U_t \) is independent of \( V_t, U_{t-1}, V_{t-1}, U_{t-2}, \ldots \) for all \( t \).
- Consequently \( \{ X_t \} \) is uncorrelated by Eq. (77) on p. 712.
- Assume furthermore that \( \{ U_t \} \) is a Gaussian stationary, uncorrelated process.
- Then \( X_t \mid I_{t-1} \sim N(\mu, V_t^2) \).

\(^a\)Engle (1982), co-winner of the 2003 Nobel Prize in Economic Sciences.
ARCH Models (continued)

- The ARCH\((p)\) process is defined by
  \[
  X_t - \mu = \left( a_0 + \sum_{i=1}^{p} a_i (X_{t-i} - \mu)^2 \right)^{1/2} U_t,
  \]
  where \(a_1, \ldots, a_p \geq 0\) and \(a_0 > 0\).

- The variance \(V_t^2\) thus satisfies
  \[
  V_t^2 = a_0 + \sum_{i=1}^{p} a_i (X_{t-i} - \mu)^2.
  \]

- The volatility at time \(t\) as estimated at time \(t-1\) depends on the \(p\) most recent observations on squared returns.

GARCH Models^{a}

- A very popular extension of the ARCH model is the generalized autoregressive conditional heteroskedastic (GARCH) process.

- The simplest GARCH\((1,1)\) process adds \(a_2 V_{t-1}^2\) to the ARCH\((1)\) process, resulting in
  \[
  V_t^2 = a_0 + a_1 (X_{t-1} - \mu)^2 + a_2 V_{t-1}^2.
  \]

- For it, \(V_t^2 = a_0 + a_1 (X_{t-1} - \mu)^2 + a_2 V_{t-1}^2\).

- The volatility at time \(t\) as estimated at time \(t-1\) depends on the squared return and the estimated volatility at time \(t-1\).

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ARCH Models (concluded)

- The ARCH\((1)\) process
  \[
  X_t - \mu = (a_0 + a_1 (X_{t-1} - \mu)^2)^{1/2} U_t
  \]
  is the simplest.

- For it,
  \[
  \text{Var}[X_t \mid X_{t-1} = x_{t-1}] = a_0 + a_1 (x_{t-1} - \mu)^2.
  \]

- The process \(\{X_t\}\) is stationary with finite variance if and only if \(a_1 < 1\), in which case \(\text{Var}[X_t] = a_0/(1 - a_1)\).

- The parameters can be estimated by statistical techniques.

GARCH Models (concluded)

- The estimate of volatility averages past squared returns by giving heavier weights to recent squared returns (see text).

- It is usually assumed that \(a_1 + a_2 < 1\) and \(a_0 > 0\), in which case the unconditional, long-run variance is given by \(a_0/(1 - a_1 - a_2)\).

- A popular special case of GARCH\((1,1)\) is the exponentially weighted moving average process, which sets \(a_0\) to zero and \(a_2\) to \(1 - a_1\).

- This model is used in J.P. Morgan’s RiskMetrics\textsuperscript{TM}.

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\(^{a}\text{Bollerslev (1986) and Taylor (1986).}\)
GARCH Option Pricing

• Options can be priced when the underlying asset’s return follows a GARCH process.

Let $S_t$ denote the asset price at date $t$.

Let $h_t^2$ be the conditional variance of the return over the period $[t, t+1]$ given the information at date $t$.

- “One day” is merely a convenient term for any elapsed time $\Delta t$.

GARCH Option Pricing (continued)

• The five unknown parameters of the model are $c, h_0, \beta_0, \beta_1, \text{ and } \beta_2$.

It is postulated that $\beta_0, \beta_1, \beta_2 \geq 0$ to make the conditional variance positive.

The above process, called the nonlinear asymmetric GARCH model, generalizes the GARCH(1, 1) model (see text).

GARCH Option Pricing (concluded)

• Adopt the following risk-neutral process for the price dynamics:

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}, \quad (78)$$

where

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon_{t+1} - c)^2, \quad (79)$$

$$\epsilon_{t+1} \sim N(0, 1) \text{ given information at date } t,$$

$r$ = daily riskless return,

$c \geq 0.$

$a$Duan (1995).

GARCH Option Pricing (concluded)

• With $y_t \equiv \ln S_t$ denoting the logarithmic price, the model becomes

$$y_{t+1} = y_t + r - \frac{h_t^2}{2} + h_t \epsilon_{t+1}. \quad (80)$$

The pair $(y_t, h_t^2)$ completely describes the current state.

The conditional mean and variance of $y_{t+1}$ are clearly

$$E[y_{t+1} | y_t, h_t^2] = y_t + r - \frac{h_t^2}{2}, \quad (81)$$

$$\text{Var}[y_{t+1} | y_t, h_t^2] = h_t^2. \quad (82)$$

$a$Duan (1995).
The Ritchken-Trevor (RT) Algorithm

- The GARCH model is a continuous-state model.
- To approximate it, we turn to trees with *discrete* states.
- Path dependence in GARCH makes the tree for asset prices explode exponentially.
- We need to mitigate this combinatorial explosion somewhat.

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The Ritchken-Trevor Algorithm (continued)

- It remains to pick the jump size and the three branching probabilities.
- The role of $\sigma$ in the Black-Scholes option pricing model is played by $h_t$ in the GARCH model.
- As a jump size proportional to $\sigma/\sqrt{n}$ is picked in the BOPM, a comparable magnitude will be chosen here.
- Define $\gamma \equiv h_0$, though other multiples of $h_0$ are possible, and
  \[
  \gamma_n \equiv \frac{\gamma}{\sqrt{n}}.
  \]
- The jump size will be some integer multiple $\eta$ of $\gamma_n$.
- We call $\eta$ the jump parameter (p. 727).

---

The Ritchken-Trevor Algorithm (continued)

- Partition a day into $n$ periods.
- Three states follow each state $(y_t, h_t^2)$ after a period.
- As the trinomial model combines, $2n + 1$ states at date $t + 1$ follow each state at date $t$ (recall p. 550).
- These $2n + 1$ values must approximate the distribution of $(y_{t+1}, h_{t+1}^2)$.
- So the conditional moments (81)–(82) at date $t + 1$ on p. 723 must be matched by the trinomial model to guarantee convergence to the continuous-state model.

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The seven values on the right approximate the distribution of logarithmic price $y_{t+1}$.
The Ritchken-Trevor Algorithm (continued)

- The middle branch does not change the underlying asset’s price.
- The probabilities for the up, middle, and down branches are

\[
    p_u = \frac{h_t^2}{2\eta^2\gamma^2} + \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}, \quad (83)
\]

\[
    p_m = 1 - \frac{h_t^2}{\eta^2\gamma^2}, \quad (84)
\]

\[
    p_d = \frac{h_t^2}{2\eta^2\gamma^2} - \frac{r - (h_t^2/2)}{2\eta\gamma\sqrt{n}}. \quad (85)
\]

- We can dispense with the intermediate nodes between dates to create a \((2n + 1)\)-nomial tree (p. 731).
- The resulting model is multinomial with \(2n + 1\) branches from any state \((y_t, h_t^2)\).
- There are two reasons behind this manipulation.
  - Interdate nodes are created merely to approximate the continuous-state model after one day.
  - Keeping the interdate nodes results in a tree that is \(n\) times as large.

It can be shown that:

- The trinomial model takes on \(2n + 1\) values at date \(t + 1\) for \(y_{t+1}\).
- These values have a matching mean for \(y_{t+1}\).
- These values have an asymptotically matching variance for \(y_{t+1}\).
- The central limit theorem thus guarantees the desired convergence as \(n\) increases.

This heptanomial tree is the outcome of the trinomial tree on p. 727 after its intermediate nodes are removed.
The Ritchken-Trevor Algorithm (continued)

- A node with logarithmic price \( y_t + \ell \eta n \) at date \( t + 1 \) follows the current node at date \( t \) with price \( y_t \) for some \(-n \leq \ell \leq n\).
- To reach that price in \( n \) periods, the number of up moves must exceed that of down moves by exactly \( \ell \).
- The probability that this happens is
  \[
  P(\ell) = \sum_{j_u,j_m,j_d} \frac{n!}{j_u!j_m!j_d!} p_u^{j_u} p_m^{j_m} p_d^{j_d},
  \]
  with \( j_u, j_m, j_d \geq 0 \), \( n = j_u + j_m + j_d \), and \( \ell = j_u - j_d \).

The Ritchken-Trevor Algorithm (continued)

- The updating rule (79) on p. 721 must be modified to account for the adoption of the discrete-state model.
- The logarithmic price \( y_t + \ell \eta n \) at date \( t + 1 \) following state \((y_t, h_t^2)\) at date \( t \) has a variance equal to
  \[
  h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 (\epsilon_{t+1} - c)^2,
  \]
  \(\epsilon_{t+1} = \frac{\ell \eta n - (r - h_t^2/2)}{h_t}, \quad \ell = 0, \pm 1, \pm 2, \ldots, \pm n,\)
  is a discrete random variable with \(2n + 1\) values.

The Ritchken-Trevor Algorithm (continued)

- Different conditional variances \( h_t^2 \) may require different \( \eta \) so that the probabilities calculated by Eqs. (83)–(85) on p. 728 lie between 0 and 1.
- This implies varying jump sizes.
- The necessary requirement \( p_m \geq 0 \) implies \( \eta \geq h_t/\gamma \).
- Hence we try
  \[
  \eta = \lceil h_t/\gamma \rceil, \lceil h_t/\gamma \rceil + 1, \lceil h_t/\gamma \rceil + 2, \ldots
  \]
  until valid probabilities are obtained or until their nonexistence is confirmed.
The Ritchken-Trevor Algorithm (continued)

- The sufficient and necessary condition for valid probabilities to exist is

\[ \left| \frac{r - (h^2/2)}{2\eta^2\sqrt{n}} \right| \leq \frac{h^2}{2\eta^2\gamma^2} \leq \min \left( 1 - \frac{|r - (h^2/2)|}{2\eta^2\sqrt{n}}, \frac{1}{2} \right). \]

- Obviously, the magnitude of \( \eta \) tends to grow with \( h_t \).
- The plot on p. 737 uses \( n = 1 \) to illustrate our points for a 3-day model.
- For example, node (1, 1) of date 1 and node (2, 3) of date 2 pick \( \eta = 2 \).

The Ritchken-Trevor Algorithm (continued)

- The topology of the tree is not a standard combining multinomial tree.
- For example, a few nodes on p. 737 such as nodes (2, 0) and (2, -1) have multiple jump sizes.
- The reason is the path dependence of the model.
  - Two paths can reach node (2, 0) from the root node, each with a different variance for the node.
  - One of the variances results in \( \eta = 1 \), whereas the other results in \( \eta = 2 \).

The Ritchken-Trevor Algorithm (concluded)

- The possible values of \( h_t^2 \) at a node are exponential nature.
- To address this problem, we record only the maximum and minimum \( h_t^2 \) at each node.\(^a\)
- Therefore, each node on the tree contains only two states \((y_t, h_{\text{max}}^2)\) and \((y_t, h_{\text{min}}^2)\).
- Each of \((y_t, h_{\text{max}}^2)\) and \((y_t, h_{\text{min}}^2)\) carries its own \( \eta \) and set of \( 2n + 1 \) branching probabilities.

\(^a\)Cakici and Topyan (2000).
Negative Aspects of the Ritchken-Trevor Algorithm

- A small $n$ may yield inaccurate option prices.
- But the tree will grow exponentially if $n$ is large enough.
  - Specifically, $n > (1 - \beta_1)/\beta_2$ when $r = c = 0$.
- A large $n$ has another serious problem: The tree cannot grow beyond a certain date.
- Thus the choice of $n$ may be limited in practice.
- The RT algorithm can be modified to be free of exponential complexity and shortened maturity.\(^\text{a}\)

\(^\text{a}\)Lyuu and Wu (2003).
\(^\text{b}\)Lyuu and Wu (2005).

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Numerical Examples

- Assume $S_0 = 100$, $y_0 = \ln S_0 = 4.60517$, $r = 0$,
  $h_0^2 = 0.0001096$, $\gamma = h_0 = 0.010469$, $n = 1$,
  $\gamma_n = \gamma/\sqrt{n} = 0.010469$, $\beta_0 = 0.000006575$, $\beta_1 = 0.9$,
  $\beta_2 = 0.04$, and $c = 0$.
- A daily variance of 0.0001096 corresponds to an annual volatility of $\sqrt{365 \times 0.0001096} \approx 20\%$.
- Let $h^2(i,j)$ denote the variance at node $(i,j)$.
- Initially, $h^2_{\text{max}}(0,0) = h^2_{\text{min}}(0,0) = h_0^2$.
- The resulting three-day tree is depicted on p. 743.

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Numerical Examples (continued)

- Let $h^2_{\text{max}}(i,j)$ denote the maximum variance at node $(i,j)$.
- Let $h^2_{\text{min}}(i,j)$ denote the minimum variance at node $(i,j)$.
- Initially, $h^2_{\text{max}}(0,0) = h^2_{\text{min}}(0,0) = h_0^2$.

Numerical Examples (continued)

Let us see how the numbers are calculated.

Start with the root node, node (0, 0).

Try \( \eta = 1 \) in Eqs. (83)–(85) on p. 728 first to obtain

\[
\begin{align*}
p_u &= 0.4974, \\
p_m &= 0, \\
p_d &= 0.5026.
\end{align*}
\]

As they are valid probabilities, the three branches from the root node use single jumps.

Numerical Examples (continued)

Move on to node (1, 1).

It has one predecessor node—node (0, 0)—and it takes an up move to reach the current node.

So apply updating rule (87) on p. 734 with \( \ell = 1 \) and \( h_2^t = h^2(0, 0) \).

The result is \( h^2(1, 1) = 0.000109645 \).

Because \( \lfloor h(1, 1)/\gamma \rfloor = 2 \), we try \( \eta = 2 \) in Eqs. (83)–(85) on p. 728 first to obtain

\[
\begin{align*}
p_u &= 0.1237, \\
p_m &= 0.7499, \\
p_d &= 0.1264.
\end{align*}
\]

As they are valid probabilities, the three branches from node (1, 1) use double jumps.
Numerical Examples (continued)

• Carry out similar calculations for node \((1, 0)\) with \(\ell = 0\) in updating rule (87) on p. 734.
• Carry out similar calculations for node \((1, -1)\) with \(\ell = -1\) in updating rule (87).
• Single jump \(\eta = 1\) works in both nodes.
• The resulting variances are
  \[
  h_2(1, 0) = 0.000105215, \\
  h_2(1, -1) = 0.000109553.
  \]

• Node \((2, 0)\) has 2 predecessor nodes, \((1, 0)\) and \((1, -1)\).
• Both have to be considered in deriving the variances.
• Let us start with node \((1, 0)\).
• Because it takes a middle move to reach the current node, we apply updating rule (87) on p. 734 with \(\ell = 0\) and \(h_2^0 = h_2^0(1, 0)\).
• The result is \(h_2^1 = 0.000101269\).

• Now move on to the other predecessor node \((1, -1)\).
• Because it takes an up move to reach the current node, apply updating rule (87) on p. 734 with \(\ell = 1\) and \(h_2^1 = h_2^1(1, -1)\).
• The result is \(h_2^2 = 0.000109603\).
• We hence record
  \[
  h_{\text{min}}^2(2, 0) = 0.000101269, \\
  h_{\text{max}}^2(2, 0) = 0.000109603.
  \]

Numerical Examples (continued)

• Consider state \(h_{\text{max}}^2(2, 0)\) first.
• Because \([h_{\text{max}}(2, 0)/\gamma] = 2\), we first try \(\eta = 2\) in Eqs. (83)–(85) on p. 728 to obtain
  \[
  p_u = 0.1237, \\
  p_m = 0.7500, \\
  p_d = 0.1263.
  \]
• As they are valid probabilities, the three branches from node \((2, 0)\) with the maximum variance use double jumps.
Numerical Examples (continued)

- Now consider state $h_{\text{min}}^2(2, 0)$.
- Because $\lfloor h_{\text{min}}(2, 0)/\gamma \rfloor = 1$, we first try $\eta = 1$ in Eqs. (83)–(85) on p. 728 to obtain
  
  \begin{align*}
  p_u &= 0.4596, \\
  p_m &= 0.0760, \\
  p_d &= 0.4644.
  \end{align*}

- As they are valid probabilities, the three branches from node $(2, 0)$ with the minimum variance use single jumps.

Numerical Examples (continued)

- Now move on to predecessor node $(1, 0)$.
- Because it also takes a down move to reach the current node, we apply updating rule (87) on p. 734 with $\ell = -1$ and $h^2_T = h^2(1, 0)$.
- The result is $h_{t+1}^2 = 0.000105609$.

Numerical Examples (continued)

- Finally, consider predecessor node $(1, -1)$.
- Because it takes a middle move to reach the current node, we apply updating rule (87) on p. 734 with $\ell = 0$ and $h^2_T = h^2(1, -1)$.
- The result is $h_{t+1}^2 = 0.000105173$.
- We hence record
  
  \begin{align*}
  h_{\text{min}}^2(2, -1) &= 0.000105173, \\
  h_{\text{max}}^2(2, -1) &= 0.0001227.
  \end{align*}
Numerical Examples (continued)

• Consider state $h_{\text{max}}^2(2, -1)$.
• Because $\lfloor h_{\text{max}}(2, -1)/\gamma \rfloor = 2$, we first try $\eta = 2$ in Eqs. (83)–(85) on p. 728 to obtain
  
  $p_u = 0.1385,$  
  $p_m = 0.7201,$  
  $p_d = 0.1414.$

• As they are valid probabilities, the three branches from node $(2, -1)$ with the maximum variance use double jumps.

Numerical Examples (continued)

• Next, consider state $h_{\text{min}}^2(2, -1)$.
• Because $\lfloor h_{\text{min}}(2, -1)/\gamma \rfloor = 1$, we first try $\eta = 1$ in Eqs. (83)–(85) on p. 728 to obtain
  
  $p_u = 0.4773,$  
  $p_m = 0.0404,$  
  $p_d = 0.4823.$

• As they are valid probabilities, the three branches from node $(2, -1)$ with the minimum variance use single jumps.

Negative Aspects of the RT Algorithm Revisited$^a$

• Recall the problems mentioned on p. 740.
• In our case, combinatorial explosion occurs when
  
  $n > \frac{1 - \beta_1}{\beta_2} = \frac{1 - 0.9}{0.04} = 2.5.$

• Suppose we are willing to accept the exponential running time and pick $n = 100$ to seek accuracy.
• But the problem of shortened maturity forces the tree to stop at date 9!

$^a$Lyuu and Wu (2003).
Backward Induction on the RT Tree

- After the RT tree is constructed, it can be used to price options by backward induction.

- Recall that each node keeps two variances $h_{max}^2$ and $h_{min}^2$.

- We now increase that number to $K$ equally spaced variances between $h_{max}^2$ and $h_{min}^2$ at each node.

- Besides the minimum and maximum variances, the other $K - 2$ variances in between are linearly interpolated.

\[ h_{min}^2(i,j) + k \frac{h_{max}^2(i,j) - h_{min}^2(i,j)}{K - 1}, \]

where $k = 0, 1, \ldots, K - 1$.

- Each interpolated variance's jump parameter and branching probabilities can be computed as before.

Backward Induction on the RT Tree (continued)

- For example, if $K = 3$, then a variance of $10.5436 \times 10^{-6}$ will be added between the maximum and minimum variances at node $(2, 0)$ on p. 743.

- In general, the $k$th variance at node $(i, j)$ is

\[ h_{min}^2(i,j) + k \frac{h_{max}^2(i,j) - h_{min}^2(i,j)}{K - 1}, \]

where $k = 0, 1, \ldots, K - 1$.

- Each interpolated variance's jump parameter and branching probabilities can be computed as before.

- In practice, log-linear interpolation works better; Lyuu and Wu (2005). Log-cubic interpolation works even better; Liu (2005).
Numerical Examples

- We next use the numerical example on p. 743 to price a European call option with a strike price of 100 and expiring at date 3.
- Recall that the riskless interest rate is zero.
- Assume \( K = 2 \); hence there are no interpolated variances.
- The pricing tree is shown on p. 765 with a call price of 0.66346.
  - The branching probabilities needed in backward induction can be found on p. 766.

Numerical Examples (continued)

- Let us derive some of the numbers on p. 765.
  - The option price for a terminal node at date 3 equals \( \max(S_3 - 100, 0) \), independent of the variance level.
- Now move on to nodes at date 2.
  - The option price at node \((2, 3)\) depends on those at nodes \((3, 5), (3, 3), \) and \((3, 1)\).
  - It therefore equals
    \[
    0.1387 \times 5.37392 + 0.7197 \times 3.19054 + 0.1416 \times 1.05240 = 3.19054.
    \]
  - Option prices for other nodes at date 2 can be computed similarly.
Numerical Examples (continued)

• For node (1, 1), the option price for both variances is
  \[0.1237 \times 3.19054 + 0.7499 \times 1.05240 + 0.1264 \times 0.14573 = 1.20241.\]

• Node (1, 0) is most interesting.
• We knew that a down move from it gives a variance of 0.000105609.
• This number falls between the minimum variance 0.000105173 and the maximum variance 0.0001227 at node (2, −1) on p. 743.

• The up move leads to the state with option price 1.05240.
• The middle move leads to the state with option price 0.48366.
• The option price at node (1, 0) is finally calculated as
  \[0.4775 \times 1.05240 + 0.0400 \times 0.48366 + 0.4825 \times 0.00362 = 0.52360.\]

Numerical Examples (continued)

• The option price corresponding to the minimum variance is 0.
• The option price corresponding to the maximum variance is 0.14573.
• The equation
  \[x \times 0.000105173 + (1 - x) \times 0.0001227 = 0.000105609\]
  is satisfied by \(x = 0.9751.\)
• So the option for the down state is approximated by
  \[x \times 0 + (1 - x) \times 0.14573 = 0.00362.\]

Numerical Examples (concluded)

• It is possible for some of the three variances following an interpolated variance to exceed the maximum variance or be exceeded by the minimum variance.
• When this happens, the option price corresponding to the maximum or minimum variance will be used during backward induction.
• An interpolated variance may choose a branch that goes into a node that is not reached in the forward-induction tree-building phase.\(^a\)

\(^a\)Lyuu and Wu (2005).