Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.
— Roger Lowenstein, *When Genius Failed*

**Terminology**
- A period denotes a unit of elapsed time.
  - Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

**Standard Notations**
The following notation will be used throughout.

$t$: a point in time.

$r(t)$: the one-period riskless rate prevailing at time $t$ for repayment one period later (the instantaneous spot rate, or short rate, at time $t$).

$P(t, T)$: the present value at time $t$ of one dollar at time $T$. 
Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period—in other words, the $(T - t)$-period spot rate at time $t$.

- The long rate is defined as $r(t, \infty)$.

$F(t, T, M)$: the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$.

$F(t, T)$: the one-period or instantaneous forward rate at time $T$ as seen at time $t$ stated on a per period basis and compounded once per period.

- It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model.
- Note that $f(t, t)$ equals the short rate $r(t)$.

Fundamental Relations

- The price of a zero-coupon bond equals
  \[ P(t, T) = \begin{cases} 
  (1 + r(t, T))^{-(T-t)} & \text{in discrete time}, \\
  e^{-r(t,T)(T-t)} & \text{in continuous time}.
  \end{cases} \]

- $r(t, T)$ as a function of $T$ defines the spot rate curve at time $t$.
- By definition,
  \[ f(t, t) = \begin{cases} 
  r(t, t + 1) & \text{in discrete time}, \\
  r(t, t) & \text{in continuous time}.
  \end{cases} \]

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:
  \[ F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \]
  - The forward price equals the future value at time $T$ of the underlying asset (see text for proof).

- Equation (93) holds whether the model is discrete-time or continuous-time, and it implies
  \[ F(t, T, M) = F(t, T, S) F(t, S, M), \quad T \leq S \leq M. \]
Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by
  \[ f(t, T, L) = \left( \frac{1}{F(t, T + T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1 \tag{94} \]

  in discrete time.

  - \( f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L)} - 1 \right) \) is the analog to Eq. (94) under simple compounding.

- In continuous time,
  \[ f(t, T, \Delta t) = \frac{\ln P(t, T + \Delta t)}{\Delta t} \to \frac{\partial \ln P(t, T)}{\partial T} \]

  by Eq. (93) on p. 845.

  - Furthermore,
  \[ f(t, T, T) = \frac{\ln(P(t, T)/P(t, T + T))}{L} \]

  is related to the spot rate curve
  \[ r(t, T) = \frac{1}{T - t} \int_t^T f(t, s) \, ds. \tag{97} \]

  • Because Eq. (96) is equivalent to
    \[ P(t, T) = e^{-\int_t^T f(t, s) \, ds}, \tag{97} \]

    the spot rate curve is
    \[ r(t, T) = \frac{1}{T - t} \int_t^T f(t, s) \, ds. \]

Fundamental Relations (continued)

- The discrete analog to Eq. (97) is
  \[ P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}. \tag{98} \]

- The short rate and the market discount function are related by
  \[ r(t) = - \frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}. \]

  - This can be verified with Eq. (96) on p. 848 and the observation that \( P(t, t) = 1 \) and \( r(t) = f(t, t) \).
Risk-Neutral Pricing

• Under the local expectations theory, the expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  
  – For all $t + 1 < T$,
  
  \[ E_t[P(t+1, T)] = 1 + r(t). \]  

  – Relation (99) in fact follows from the risk-neutral valuation principle, Theorem 14 (p. 412).

Risk-Neutral Pricing (continued)

• The local expectations theory is thus a consequence of the existence of a risk-neutral probability $\pi$.

• Rewrite Eq. (99) as

  \[ \frac{E_\pi[P(t+1, T)]}{1 + r(t)} = P(t, T). \]  

  – It says the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate.

Risk-Neutral Pricing (continued)

• Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ \frac{P(t+1, T)}{1 + r(t)} \right] = E_t^\pi \left[ \frac{E_{t+1}^\pi[P(t+2, T)]}{(1 + r(t))(1 + r(t+1))} \right] = \cdots = E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t+1)) \cdots (1 + r(T-1))} \right]. \]  

(100)

Risk-Neutral Pricing (concluded)

• Equation (99) on p. 850 can also be expressed as

\[ E_t[P(t+1, T)] = F(t, t+1, T). \]

• Hence the forward price for the next period is an unbiased estimator of the expected bond price.
Continuous-Time Risk-Neutral Pricing

- In continuous time, the local expectations theory implies
  \[ P(t, T) = E_t \left[ e^{- \int_t^T r(s) \, ds} \right], \quad t < T. \]  \hspace{1cm} (101)
- Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which denotes the rolled-over money market account.
- When the local expectations theory holds, riskless arbitrage opportunities are impossible.

Interest Rate Swaps

- Consider an interest rate swap made at time \( t \) with payments to be exchanged at times \( t_1, t_2, \ldots, t_n \).
- The fixed rate is \( c \) per annum.
- The floating-rate payments are based on the future annual rates \( f_0, f_1, \ldots, f_{n-1} \) at times \( t_0, t_1, \ldots, t_{n-1} \).
- For simplicity, assume \( t_{i+1} - t_i \) is a fixed constant \( \Delta t \) for all \( i \), and the notional principal is one dollar.
- If \( t < t_0 \), we have a forward interest rate swap.
- The ordinary swap corresponds to \( t = t_0 \).

Interest Rate Swaps (continued)

- The amount to be paid out at time \( t_{i+1} \) is \( (f_i - c) \Delta t \) for the floating-rate payer.
  - Simple rates are adopted here.
- Hence \( f_i \) satisfies
  \[ P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}. \]
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.

Swap Rate

- The swap rate, which gives the swap zero value, equals
  \[ S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^n P(t, t_i) \Delta t}. \] (102)

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.
- For an ordinary swap, \( P(t, t_0) = 1 \).

The Binomial Model

- The analytical framework can be nicely illustrated with the binomial model.
- Suppose the bond price \( P \) can move with probability \( q \) to \( Pu \) and probability \( 1 - q \) to \( Pd \), where \( u > d \):

\[
\begin{array}{c}
Pd \\
P \\
q \\nPu
\end{array}
\]

The Binomial Model (continued)

- Over the period, the bond’s expected rate of return is
  \[ \hat{\mu} \equiv \frac{qPu + (1 - q)Pd}{P} - 1 = qu + (1 - q)d - 1. \] (103)

- The variance of that return rate is
  \[ \sigma^2 \equiv q(1 - q)(u - d)^2. \] (104)

- The bond whose maturity is only one period away will move from a price of \( 1/(1 + r) \) to its par value \$1.
- This is the money market account modeled by the short rate.
The Binomial Model (continued)

- The market price of risk is defined as $\lambda \equiv (\hat{\mu} - r) / \hat{\sigma}$.
- The same arbitrage argument as in the continuous-time case can be employed to show that $\lambda$ is independent of the maturity of the bond (see text).

The Binomial Model (concluded)

- Now change the probability from $q$ to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1 + r) - d}{u - d}, \quad (105)$$

which is independent of bond maturity and $q$.

- Recall the BOPM.

- The bond’s expected rate of return becomes

$$\frac{pP_u + (1-p)P_d}{P} - 1 = pu + (1-p)d - 1 = r.$$ 

- The local expectations theory hence holds under the new probability measure $p$.

Numerical Examples

- Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

4% $\leftarrow$ 8%   

2%

- No real-world probabilities are specified.

- The prices of one- and two-year zero-coupon bonds are, respectively,

$$100/1.04 = 96.154, \quad 100/(1.05)^2 = 90.703.$$ 

- They follow the binomial processes on p. 866.
The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.

The pricing of derivatives can be simplified by assuming investors are risk-neutral.

Suppose all securities have the same expected one-period rate of return, the riskless rate.

Then

\[(1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4%,\]

where \(p\) denotes the risk-neutral probability of an up move in rates.

Solving the equation leads to \(p = 0.319\).

Interest rate contingent claims can be priced under this probability.

A one-year European call on the two-year zero with a $95 strike price has the payoffs,

\[C \leftarrow 0.000\]

\[3.039\]

To solve for the option value \(C\), we replicate the call by a portfolio of \(x\) one-year and \(y\) two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

• This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

• They give \( x = -0.5167 \) and \( y = 0.5580 \).

• Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.

Numerical Examples: Fixed-Income Options (concluded)

• An equivalent method is to utilize risk-neutral pricing.

• The above call option is worth

\[ C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]

the same as before.

• This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

Numerical Examples: Fixed-Income Options (continued)

• This price is derived without assuming any version of an expectations theory.

• Instead, the arbitrage-free price is derived by replication.

• The price of an interest rate contingent claim does not depend directly on the real-world probabilities.

• The dependence holds only indirectly via the current bond prices.

Numerical Examples: Futures and Forward Prices

• A one-year futures contract on the one-year rate has a payoff of \( 100 - r \), where \( r \) is the one-year rate at maturity, as shown below.

\[ F = \begin{cases} 92 = 100 - 8 \\ 98 = 100 - 2 \end{cases} \]

• As the futures price \( F \) is the expected future payoff (see text), \( F = (1 - p) \times 92 + p \times 98 = 93.914 \).

• On the other hand, the forward price for a one-year forward contract on a one-year zero-coupon bond equals \( 90.703/96.154 = 94.331\% \).

• The forward price exceeds the futures price.
**Numerical Examples: Mortgage-Backed Securities**

- Consider a 5%-coupon, two-year mortgage-backed security without amortization, prepayments, and default risk.
- Its cash flow and price process are illustrated on p. 875.
- Its fair price is
  \[ M = \frac{(1 - p) \times 102.222 + p \times 107.941}{1.04} = 100.045. \]
- Identical results could have been obtained via arbitrage considerations.

**Numerical Examples: MBSs (continued)**

- Suppose that the security can be prepaid at par.
- It will be prepaid only when its price is higher than par.
- Prepayment will hence occur only in the “down” state when the security is worth 102.941 (excluding coupon).
- The price therefore follows the process,
  \[ M = \frac{(1 - p) \times 102.222 + p \times 105}{1.04} = 99.142. \]
- The security is worth
  \[ M = \frac{(1 - p) \times 102.222 + p \times 105}{1.04} = 99.142. \]

The left diagram depicts the cash flow; the right diagram illustrates the price process.

**Numerical Examples: MBSs (continued)**

- The cash flow of the principal-only (PO) strip comes from the mortgage’s principal cash flow.
- The cash flow of the interest-only (IO) strip comes from the interest cash flow (p. 878(a)).
- Their prices hence follow the processes on p. 878(b).
- The fair prices are
  \[ PO = \frac{(1 - p) \times 92.593 + p \times 100}{1.04} = 91.304, \]
  \[ IO = \frac{(1 - p) \times 9.630 + p \times 5}{1.04} = 7.839. \]
The price 9.630 is derived from $5 + (5/1.08)$.

### Numerical Examples: MBSs (continued)

- Suppose the mortgage is split into half floater and half inverse floater.
- Let the floater (FLT) receive the one-year rate.
- Then the inverse floater (INV) must have a coupon rate of
  \[(10\% - \text{one-year rate})\]
  to make the overall coupon rate 5%.
- Their cash flows as percentages of par and values are shown on p. 880.

### Numerical Examples: MBSs (concluded)

- On p. 880, the floater’s price in the up node, 104, is derived from $4 + (108/1.08)$.
- The inverse floater’s price 100.444 is derived from $6 + (102/1.08)$.
- The current prices are
  \[
  \text{FLT} = \frac{1}{2} \times \frac{104}{1.04} = 50,
  \]
  \[
  \text{INV} = \frac{1}{2} \times \frac{(1 - p) \times 100.444 + p \times 106}{1.04} = 49.142.
  \]
8. What’s your problem? Any moron can understand bond pricing models. — *Top Ten Lies Finance Professors Tell Their Students*

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**Introduction**

- This chapter surveys equilibrium models.
- Since the spot rates satisfy
  \[
  r(t,T) = -\frac{\ln P(t,T)}{T-t},
  \]
  the discount function \( P(t,T) \) suffices to establish the spot rate curve.
- All models to follow are short rate models.
- Unless stated otherwise, the processes are risk-neutral.

---

**The Vasicek Model\(^a\)**

- The short rate follows
  \[
  dr = \beta(\mu - r) \, dt + \sigma \, dW.
  \]
- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).
- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma \, dW \).
- Since the process is an Ornstein-Uhlenbeck process,
  \[
  E[r(T) | r(t) = r] = \mu + (r - \mu) e^{-\beta(T-t)}
  \]
  from Eq. (53) on p. 467.

\(^a\)Vasicek (1977).
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[ P(t, T) = A(t, T) e^{-B(t, T) r(t)}, \]

where

\[ A(t, T) = \begin{cases} \exp \left[ \frac{(B(t, T) - T + 1)(3a^2 - a^2/2) - \sigma^2 B(t, T)^2}{3\beta^2} \right] & \text{if } \beta \neq 0, \\ \exp \left[ \frac{\sigma^2 (T - t)^3}{6} \right] & \text{if } \beta = 0. \end{cases} \]

and

\[ B(t, T) = \begin{cases} \frac{1 - e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\ T - t & \text{if } \beta = 0. \end{cases} \]

The Vasicek Model (concluded)

- If \( \beta = 0 \), then \( P \) goes to infinity as \( T \to \infty \).
- Sensibly, \( P \) goes to zero as \( T \to \infty \) if \( \beta \neq 0 \).
- Even if \( \beta \neq 0 \), \( P \) may exceed one for a finite \( T \).
- The spot rate volatility structure is the curve \((\partial r(t, T)) / (\partial r) \sigma = \sigma B(t, T) / (T - t)\).
- When \( \beta > 0 \), the curve tends to decline with maturity.
- The speed of mean reversion, \( \beta \), controls the shape of the curve; indeed, higher \( \beta \) leads to greater attenuation of volatility with maturity.

The Vasicek Model: Options on Zeros

- Consider a European call with strike price \( X \) expiring at time \( T \) on a zero-coupon bond with par value $1 and maturing at time \( s > T \).
- Its price is given by

\[ P(t, s) N(x) - X P(t, T) N(x - \sigma_v). \]

\(^a\)Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

- Above

\[ x \equiv \frac{1}{\sigma_v} \ln \left( \frac{P(t, s)}{P(t, T) X} \right) + \frac{\sigma_v}{2}, \]
\[ \sigma_v \equiv v(t, T) B(T, s), \]
\[ v(t, T)^2 \equiv \begin{cases} \sigma^2 \left[ 1 - e^{-2\beta(T-t)} \right], & \text{if } \beta \neq 0 \\ \sigma^2(T-t), & \text{if } \beta = 0. \end{cases} \]

- By the put-call parity, the price of a European put is

\[ XP(t, T) \Phi(-x + \sigma_v) - P(t, s) \Phi(-x). \]

Binomial Vasicek

- Consider a binomial model for the short rate in the time interval \([0, T]\) divided into \(n\) identical pieces.
- Let \(\Delta t \equiv T/n\) and

\[ p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r)}{2\sigma} \sqrt{\Delta t}. \]

- The following binomial model converges to the Vasicek model,\(^a\)

\[ r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n. \]

\(^a\)Nelson and Ramaswamy (1990).

Binomial Vasicek (continued)

- Above, \(\xi(k) = \pm 1\) with

\[ \text{Prob}[\xi(k) = 1] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}. \]

- Observe that the probability of an up move, \(p\), is a decreasing function of the interest rate \(r\).
- This is consistent with mean reversion.

Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its constant volatility, \(\sigma\).
- For a general process \(Y\) with nonconstant volatility, the resulting binomial tree may not combine.
The Cox-Ingersoll-Ross Model

- It is the following square-root short rate model:
  \[ dr = \beta(\mu - r) \, dt + \sigma \sqrt{r} \, dW. \]  
  (107)
- The diffusion differs from the Vasicek model by a multiplicative factor \( \sqrt{r} \).
- The parameter \( \beta \) determines the speed of adjustment.
- The short rate can reach zero only if \( 2\beta \mu < \sigma^2 \).
- See text for the bond pricing formula.

\footnote{Cox, Ingersoll, and Ross (1985).}

Binomial CIR (continued)

- Instead, consider the transformed process
  \[ x(r) \equiv 2\sqrt{r}/\sigma. \]
- It follows
  \[ dx = m(x) \, dt + dW, \]
  where
  \[ m(x) \equiv 2\beta \mu/(\sigma^2 x) - (\beta x/2) - 1/(2x). \]
- Since this new process has a constant volatility, its associated binomial tree combines.

Binomial CIR

- We want to approximate the short rate process in the time interval \([0, T]\).
- Divide it into \( n \) periods of duration \( \Delta t \equiv T/n \).
- Assume \( \mu, \beta \geq 0 \).
- A direct discretization of the process is problematic because the resulting binomial tree will not combine.

Binomial CIR (continued)

- Construct the combining tree for \( r \) as follows.
- First, construct a tree for \( x \).
- Then transform each node of the tree into one for \( r \) via the inverse transformation \( r = f(x) \equiv x^2\sigma^2/4 \) (p. 898).
Binomial CIR (concluded)

- The probability of an up move at each node $r$ is
  \[ p(r) = \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}. \]

- $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.
- $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

- Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.

Numerical Examples

- Consider the process,
  \[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]
  for the time interval $[0, 1]$ given the initial rate $r(0) = 0.04$.

- We shall use $\Delta t = 0.2$ (year) for the binomial approximation.

- See p. 901(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (continued)

- Consider the node which is the result of an up move from the root.
- Since the root has \( x = 2\sqrt{r(0)/\sigma} = 4 \), this particular node’s \( x \) value equals \( 4 + \sqrt{\Delta t} = 4.4472135955 \).
- Use the inverse transformation to obtain the short rate \( x^2 \times (0.1)^2/4 \approx 0.0494442719102 \).

A General Method for Constructing Binomial Models

- We are given a continuous-time process \( dy = \alpha(y, t) dt + \sigma(y, t) dW \).
- Make sure the binomial model’s drift and diffusion converge to the above process by setting the probability of an up move to \( \frac{\alpha(y, t) \Delta t + y - y_u}{y_u - y_d} \).
- Here \( y_u \equiv y + \sigma(y, t)\sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t)\sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).
- The displacements are identical, at \( \sigma(y, t)\sqrt{\Delta t} \).

A General Method (continued)

- But the binomial tree may not combine: \( \sigma(y, t)\sqrt{\Delta t} - \sigma(y_u, t)\sqrt{\Delta t} \neq -\sigma(y, t)\sqrt{\Delta t} + \sigma(y_d, t)\sqrt{\Delta t} \) in general.
- When \( \sigma(y, t) \) is a constant independent of \( y \), equality holds and the tree combines.
- To achieve this, define the transformation \( x(y, t) \equiv \int^y \sigma(z, t)^{-1} dz \).
- Then \( x \) follows \( dx = m(y, t) dt + dW \) for some \( m(y, t) \) (see text).
A General Method (continued)

- The key is that the diffusion term is now a constant, and the binomial tree for $x$ combines.
- The probability of an up move remains
  \[ \alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t) \]
  \[ y_u(x, t) - y_d(x, t), \]
  where $y(x, t)$ is the inverse transformation of $x(y, t)$ from $x$ back to $y$.
- Note that $y_u(x, t) \equiv y(x + \sqrt{\Delta t}, t + \Delta t)$ and $y_d(x, t) \equiv y(x - \sqrt{\Delta t}, t + \Delta t)$.

On One-Factor Short Rate Models

- By using only the short rate, they ignore other rates on the yield curve.
- Such models also restrict the volatility to be a function of interest rate levels only.
- The prices of all bonds move in the same direction at the same time (their magnitudes may differ).
- The returns on all bonds thus become highly correlated.
- In reality, there seems to be a certain amount of independence between short- and long-term rates.

A General Method (concluded)

- The transformation is
  \[ \int r^r (\sigma \sqrt{z})^{-1} dz = 2\sqrt{r}/\sigma \]
  for the CIR model.
- The transformation is
  \[ \int S^S (\sigma z)^{-1} dz = (1/\sigma) \ln S \]
  for the Black-Scholes model.
- The familiar binomial option pricing model in fact discretizes $\ln S$ not $S$.

On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two-factor ones.

Options on Coupon Bonds

The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.

Consider a European call expiring at time $T$ on a bond with par value $1$.

Let $X$ denote the strike price.

The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all $i$.

The payoff for the option is

$$\max \left( \sum_{i=1}^{n} c_i P(r(T), T, t_i) - X, 0 \right).$$

Options on Coupon Bonds (continued)

- At time $T$, there is a unique value $r^*$ for $r(T)$ that renders the coupon bond’s price equal the strike price $X$.
- This $r^*$ can be obtained by solving $X = \sum_{i=1}^{n} c_i P(r(T, t_i))$ numerically for $r$.
- The solution is also unique for one-factor models whose bond price is a monotonically decreasing function of $r$.
- Let $X_i \equiv P(r^*, T, t_i)$, the value at time $T$ of a zero-coupon bond with par value $1$ and maturing at time $t_i$ if $r(T) = r^*$.

Options on Coupon Bonds (concluded)

- Note that $P(r(T), T, t_i) \geq X_i$ if and only if $r(T) \leq r^*$.
- As $X = \sum_i c_i X_i$, the option’s payoff equals

$$\sum_{i=1}^{n} c_i \times \max(P(r(T), T, t_i) - X_i, 0).$$

- Thus the call is a package of $n$ options on the underlying zero-coupon bond.

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