General Derivatives Pricing

- In general the underlying asset \( S \) may not be traded.
  - Interest rate, for instance, is not a traded security.
- Let \( S \) follow the Ito process \( \frac{dS}{S} = \mu dt + \sigma dW \), where \( \mu \) and \( \sigma \) may depend only on \( S \) and \( t \).
- Let \( f_1(S,t) \) and \( f_2(S,t) \) be the prices of two derivatives with dynamics \( \frac{df_i}{f_i} = \mu_i dt + \sigma_i dW \), \( i = 1, 2 \).
  - They share the same Wiener process as \( S \).

A portfolio consisting of \( \sigma_2 f_2 \) units of the first derivative and \( -\sigma_1 f_1 \) units of the second derivative is instantaneously riskless:

\[
\sigma_2 f_2 \frac{df_1}{f_1} - \sigma_1 f_1 \frac{df_2}{f_2} = \sigma_2 f_2 f_1 (\mu_1 dt + \sigma_1 dW) - \sigma_1 f_1 f_2 (\mu_2 dt + \sigma_2 dW) = (\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2) dt.
\]

Therefore,

\[
(\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2) dt = r(\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2) dt,
\]

or \( \sigma_2 \mu_1 - \sigma_1 \mu_2 = r(\sigma_2 - \sigma_1) \).

General Derivatives Pricing (continued)

- After rearranging the terms,
  \[
  \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda \text{ for some } \lambda.
  \]
- A derivative whose value depends only on \( S \) and \( t \) and which follows the Ito process \( \frac{df}{f} = \mu dt + \sigma dW \) must thus satisfy
  \[
  \frac{\mu - r}{\sigma} = \lambda \text{ or } \mu = r + \lambda \sigma. \quad (59)
  \]
- We call \( \lambda \) the market price of risk, which is independent of the specifics of the derivative.

General Derivatives Pricing (continued)

- Ito's lemma can be used to derive the formulas for \( \mu \) and \( \sigma \):
  \[
  \mu = \frac{1}{f} \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right),
  \quad \sigma = \frac{\sigma S}{f} \frac{\partial f}{\partial S}.
  \]
- Substitute the above into Eq. (59) on p. 509 to obtain
  \[
  \frac{\partial f}{\partial t} + (\mu - \lambda \sigma) S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \quad (60)
  \]
General Derivatives Pricing (concluded)

- The presence of $\mu$ shows that the investor’s risk preference is relevant.
- The derivative may be dependent on the underlying asset’s growth rate and the market price of risk.
- Only when the underlying variable is the price of a traded security can we assume $\mu = r$ in pricing.
  - Note that in such a case, $\lambda = 0$.

When Professors Scholes and Merton and I invested in warrants, Professor Merton lost the most money.
And I lost the least.
— Fischer Black (1938–1995)

Hedging

Delta Hedge

- The delta (hedge ratio) of a derivative $f$ is defined as $\Delta = \partial f / \partial S$.
- Thus $\Delta f \approx \Delta \times \Delta S$ for relatively small changes in the stock price, $\Delta S$.
- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.
- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.
Delta Hedge (concluded)

- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.
- This was the gist of the Black-Scholes-Merton argument.

Implementing Delta Hedge

- We want to hedge $N$ short derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus $B$ borrowed dollars such that
  $$-N \times f + N \times \Delta \times S - B = 0.$$
- At next rebalancing point when the delta is $\Delta'$, buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of
  $$B' = N \times \Delta' \times S' - N \times f'.$$
- Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.

Example

- A hedger is short 10,000 European calls.
- $\sigma = 30\%$ and $r = 6\%$.
- This call’s expiration is four weeks away, its strike price is $50$, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1,000,000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated well if the cumulative cost of trading stock is close to the call premium’s FV.

Example (continued)

- As $\Delta = 0.538560$, $N \times \Delta = 538,560$ shares are purchased for a total cost of $538,560 \times 50 = 26,928,000$ dollars to make the portfolio delta-neutral.
- The trader finances the purchase by borrowing
  $$B = N \times \Delta \times S - N \times f = 25,160,090$$
dollars net.
- The portfolio has zero net value now.
Example (continued)

- At 3 weeks to expiration, the stock price rises to $51.
- The new call value is $f' = 2.10580$.
- So the portfolio is worth
  
  \[-N \times f' + 538,560 \times 51 - Be^{0.06/52} = 171,622\]
  
  before rebalancing.

Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta $\Delta' = 0.640355$, the trader buys
  
  \[N \times (\Delta' - \Delta) = 101,795\]
  
  shares for $5,191,545$.
- The number of shares is increased to $N \times \Delta' = 640,355$.

Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as $171,622$ can be withdrawn.
- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.
- In fact, the tracking error is positive about 68% of the time even though its expected value is essentially zero.\(^a\)
- It is furthermore proportional to vega.

\(^a\)Boyle and Emanuel (1980).
The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).

Example (concluded)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for $50,000,000.
- The trader is left with an obligation of
  \[51,524,853 - 50,000,000 = 1,524,853,\]
  which represents the replication cost.
- Compared with the FV of the call premium,
  \[1,767,910 \times e^{0.06 \times 4/52} = 1,776,088,\]
the net gain is \(1,776,088 - 1,524,853 = 251,235.\)

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, \(\Delta f\), due to changes in the stock price, \(\Delta S\).
- When \(\Delta S\) is not small, the second-order term, gamma
  \[\Gamma \equiv \frac{\partial^2 f}{\partial S^2},\]
  helps (theoretically).
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.
- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call \(f_2\) is brought in.
- To set up a delta-gamma hedge, we solve
  \[-N \times f + n_1 \times S + n_2 \times f_2 - B = 0 \quad \text{(self-financing)},\]
  \[-N \times \Delta + n_1 \times n_2 \times \Delta_2 - 0 = 0 \quad \text{(delta neutrality)},\]
  \[-N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 = 0 \quad \text{(gamma neutrality)},\]
for \(n_1, n_2,\) and \(B.\)
- The gammas of the stock and bond are 0.
Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, A Day in the Life (1967)

Trees

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 227.
  - It cannot apply to American options.
- We will now apply it to price barrier options.
The Reflection Principle

• Imagine a particle at position $(0, -a)$ on the integral lattice that is to reach $(n, -b)$.
• Without loss of generality, assume $a > 0$ and $b \geq 0$.
• This particle’s movement:
  - $(i, j) \rightarrow (i + 1, j + 1)$ up move $S \rightarrow Su$
  - $(i, j) \rightarrow (i + 1, j - 1)$ down move $S \rightarrow Sd$
• How many paths touch the $x$ axis?

---

The Reflection Principle (continued)

• For a path from $(0, -a)$ to $(n, -b)$ that touches the $x$ axis, let $J$ denote the first point this happens.
• Reflect the portion of the path from $(0, -a)$ to $J$.
• A path from $(0, a)$ to $(n, -b)$ is constructed.
• It also hits the $x$ axis at $J$ for the first time (see figure next page).
• The one-to-one mapping shows the number of paths from $(0, -a)$ to $(n, -b)$ that touch the $x$ axis equals the number of paths from $(0, a)$ to $(n, -b)$.

---

The Reflection Principle (concluded)

• A path of this kind has $(n + b + a)/2$ down moves and $(n - b - a)/2$ up moves.
• Hence there are

$$
\binom{n}{\frac{n + a + b}{2}}
$$

such paths for even $n + a + b$.

- Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$. 
Pricing Barrier Options (Lyuu, 1998)

- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define
  
  $a = \left\lfloor \frac{\ln (X / (S d^n))}{\ln (u / d)} \right\rfloor = \left[ \frac{\ln(X) - \ln(S)}{2 \sigma \sqrt{t}} + \frac{n}{2} \right]$,
  
  $h = \left\lfloor \frac{\ln (H / (S d^n))}{\ln (u / d)} \right\rfloor = \left[ \frac{\ln(H) - \ln(S)}{2 \sigma \sqrt{t}} + \frac{n}{2} \right]$.

  - $h$ is such that $\tilde{H} = S u^h d^{n-h}$ is the terminal price that is closest to, but does not exceed $H$.
  - $a$ is such that $\tilde{X} = S u^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by $X$.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with $n$ moves hence ends up in the money if and only if the number of up moves is at least $a$.
- The price $S u^b d^{n-k}$ is at a distance of $2k$ from the lowest possible price $S d^n$ on the binomial tree.
  
  $S u^b d^{n-k} = S d^{-k} d^{n-k} = S d^{n-2k}$.

Pricing Barrier Options (continued)

- The number of paths from $S$ to the terminal price $S u^b d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j (1-p)^{n-j}$.
- With reference to p. 537, the reflection principle can be applied with $a = n - 2h$ and $b = 2j - 2h$ in Eq. (61) on p. 534 by treating the $S$ line as the $x$ axis.
- Therefore,
  
  $\binom{n}{n+(n-2h)+(2j-2h)} = \binom{n}{n-2h+j}$

  paths hit $\tilde{H}$ in the process for $h \leq n/2$. 
Pricing Barrier Options (concluded)

- The terminal price $S u^j d^{n-j}$ is reached by a path that hits the effective barrier with probability
  \[
  \binom{n}{n - 2h + j} p^j (1 - p)^{n-j}.
  \]
- The option value equals
  \[
  R^n \sum_{j=a}^{2h} \binom{n}{n - 2h + j} p^j (1 - p)^{n-j} (S u^j d^{n-j} - X).
  \]
- $R \equiv e^{r \tau / n}$ is the riskless return per period.
- It implies a linear-time algorithm.

Convergence of BOPM (continued)

- Convergence is actually good if we limit $n$ to certain values—191, for example.
- These values make the true barrier coincide with or occur just above one of the stock price levels, that is, $H \approx S d^j = S e^{-j \sigma \sqrt{\tau / n}}$ for some integer $j$.
- The preferred $n$’s are thus
  \[
  n = \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor, \quad j = 1, 2, 3, \ldots
  \]
- There is only one minor technicality left.

Convergence of BOPM

- Equation (63) results in the sawtooth-like convergence shown on p. 299.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.
Convergence of BOPM (concluded)

- The preferred \( n \)'s are now

\[
\ell \quad \text{if } \ell - j \text{ is even} \\
\ell - 1 \quad \text{otherwise}
\]

\[
j = 1, 2, 3, \ldots,
\]

where

\[
\ell \equiv \left\lfloor \frac{\tau}{\ln(S/H) / (j\sigma)^2} \right\rfloor.
\]

- So evaluate pricing formula (63) on p. 539 only with the \( n \)'s above.

Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large \( n \)'s (p. 546).

- This has profound consequences.

- For example, pricing is prohibitively time consuming when \( S \approx H \) because \( n \sim 1/\ln^2(S/H) \).

- This observation is indeed true of standard quadratic-time binomial tree algorithms.

- But it no longer applies to linear-time algorithms (p. 547).
Trinomial Tree

• Set up a trinomial approximation to the geometric Brownian motion \( dS/S = r \Delta t + \sigma dW \).\(^a\)

• The three stock prices at time \( \Delta t \) are \( S, Su, \) and \( Sd \), where \( ud = 1 \).

• Impose the matching of mean and that of variance:

\[
\begin{align*}
1 &= p_u + p_m + p_d, \\
SM &\equiv (p_uS + p_m + (p_d/u))S, \\
S^2V &\equiv p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2.
\end{align*}
\]

\(^a\)Boyle (1988).
Trinomial Tree (continued)

- Use linear algebra to verify that
  \[ p_u = \frac{u (V + M^2 - M) - (M - 1)}{(u - 1) (u^2 - 1)}, \]
  \[ p_d = \frac{u^2 (V + M^2 - M) - u^3 (M - 1)}{(u - 1) (u^2 - 1)}. \]
  - In practice, must make sure the probabilities lie between 0 and 1.
- Countless variations.

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting \( \lambda \) so that the barrier is hit exactly.\(^a\)
- It takes
  \[ h = \frac{\ln(S/H)}{\lambda \sigma \sqrt{\Delta t}} \]
  consecutive down moves to go from \( S \) to \( H \) if \( h \) is an integer, which is easy to achieve by adjusting \( \lambda \).

\(^a\)Ritchken (1995).

Trinomial Tree (concluded)

- Use \( u = e^{\lambda \sigma \sqrt{\Delta t}} \), where \( \lambda \geq 1 \) is a tunable parameter.
- Then
  \[ p_u \rightarrow \frac{1}{2 \lambda^2} + \frac{(r + \sigma^2) \sqrt{\Delta t}}{2 \lambda \sigma}, \]
  \[ p_d \rightarrow \frac{1}{2 \lambda^2} - \frac{(r - 2 \sigma^2) \sqrt{\Delta t}}{2 \lambda \sigma}. \]
- A nice choice for \( \lambda \) is \( \sqrt{\pi/2} \).\(^a\)

\(^a\)Omberg (1988).

Barrier Options Revisited (continued)

- Typically, we find the smallest \( \lambda \geq 1 \) such that \( h \) is an integer.
- That is, we find the largest integer \( j \geq 1 \) that satisfies
  \[ \frac{\ln(S/H)}{j \sigma \sqrt{\Delta t}} \geq 1 \]
  and then let
  \[ \lambda = \frac{\ln(S/H)}{j \sigma \sqrt{\Delta t}}. \]
  - Such a \( \lambda \) may not exist for very small \( n \)'s.
  - This is not hard to check.
- This done, one of the layers of the trinomial tree coincides with the barrier.
Barrier Options Revisited (concluded)

- The following probabilities may be used,

\[
p_u = \frac{1}{2\lambda^2} + \frac{\mu'\sqrt{\Delta t}}{2\lambda \sigma},
\]
\[
p_m = 1 - \frac{1}{\lambda^2},
\]
\[
p_d = \frac{1}{2\lambda^2} - \frac{\mu'\sqrt{\Delta t}}{2\lambda \sigma} - \mu' \equiv r - \sigma^2/2.
\]