Geometric Brownian Motion

- Consider the geometric Brownian motion process
  \( Y(t) \equiv e^{X(t)} \)
  - \( X(t) \) is a \((\mu, \sigma)\) Brownian motion.
- As \( \partial Y / \partial X = Y \) and \( \partial^2 Y / \partial X^2 = Y \), Ito's formula (51) on p. 453 implies
  \[
  \frac{dY}{Y} = \left( \mu + \frac{\sigma^2}{2} \right) dt + \sigma dW.
  \]
- The annualized instantaneous rate of return is \( \mu + \frac{\sigma^2}{2} \) not \( \mu \).

Product of Geometric Brownian Motion Processes (continued)

- The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion.
- Note that
  \[
  Y = \exp \left[ \left( a - \frac{b^2}{2} \right) dt + b dW_Y \right], \\
  Z = \exp \left[ \left( f - \frac{g^2}{2} \right) dt + g dW_Z \right], \\
  U = \exp \left[ \left( a + f - \frac{(b^2 + g^2)}{2} \right) dt + b dW_Y + g dW_Z \right].
  \]
Quotients of Geometric Brownian Motion Processes
• Suppose $Y$ and $Z$ are drawn from p. 460.
• Let $U \equiv Y/Z$.
• We now show that
  \[
  \frac{dU}{U} = (a - f + g^2 - bg\rho) \, dt + b \, dW_Y - g \, dW_Z.
  \]
  \[\text{(52)}\]
• Keep in mind that $dW_Y$ and $dW_Z$ have correlation $\rho$.

Quotients of Geometric Brownian Motion Processes (concluded)
• The multidimensional Ito’s lemma (Theorem 18 on p. 457) can be employed to show that
  \[
  dU = \left( \frac{1}{Z} \right) dY - \left( \frac{1}{Z^2} \right) dZ - \left( \frac{1}{Z^2} \right) dY \, dZ + (Y/Z^2) \, (dZ)^2
  \]
  \[= (a - f + g^2 - bg\rho) \, dt + b \, dW_Y - g \, dW_Z.
  \]

Ornstein-Uhlenbeck Process
• The Ornstein-Uhlenbeck process:
  \[
  dX = -\kappa X \, dt + \sigma \, dW,
  \]
  where $\kappa, \sigma \geq 0$.
• It is known that
  \[
  E[X(t)] = e^{-\kappa(t-t_0)} E[x_0],
  \]
  \[
  \text{Var}[X(t)] = \frac{\sigma^2}{2\kappa} \left( 1 - e^{-2\kappa(t-t_0)} \right) + e^{-2\kappa(t-t_0)} \text{Var}[x_0],
  \]
  \[
  \text{Cov}[X(s), X(t)] = \frac{\sigma^2}{2\kappa} \left( 1 - e^{-2\kappa(s-t)} \right) + e^{-2\kappa(t-s)} \text{Var}[x_0],
  \]
  for $t_0 \leq s \leq t$ and $X(t_0) = x_0$.

Ornstein-Uhlenbeck Process (continued)
• $X(t)$ is normally distributed if $x_0$ is a constant or normally distributed.
• $X$ is said to be a normal process.
• $E[x_0] = x_0$ and $\text{Var}[x_0] = 0$ if $x_0$ is a constant.
• The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When $X > 0$, $X$ is pulled $X$ toward zero.
  - When $X < 0$, it is pulled toward zero again.
Ornstein-Uhlenbeck Process (continued)

- Another version:
  \[
  dX = \kappa(\mu - X) \, dt + \sigma \, dW,
  \]
  where \( \sigma \geq 0 \).

- Given \( X(t_0) = x_0 \), a constant, it is known that
  \[
  E[X(t)] = \mu + (x_0 - \mu) e^{-\kappa(t-t_0)}, \quad (53)
  \]
  \[
  \text{Var}[X(t)] = \frac{\sigma^2}{2\kappa} \left[ 1 - e^{-2\kappa(t-t_0)} \right],
  \]
  for \( t_0 \leq t \).

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly \( \mu \) and \( \sigma/\sqrt{2\kappa} \), respectively.

- For large \( t \), the probability of \( X < 0 \) is extremely unlikely in any finite time interval when \( \mu > 0 \) is large relative to \( \sigma/\sqrt{2\kappa} \) (say \( \mu > 4\sigma/\sqrt{2\kappa} \)).

- The process is mean-reverting.
  - \( X \) tends to move toward \( \mu \).
  - Useful for modeling term structure, stock price volatility, and stock price return.

Interest Rate Models

- Suppose the short rate \( r \) follows process
  \[
  dr = \mu(r,t) \, dt + \sigma(r,t) \, dW.
  \]

- Let \( P(r,t,T) \) denote the price at time \( t \) of a zero-coupon bond that pays one dollar at time \( T \).

- Write its dynamics as
  \[
  \frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.
  \]

  - The expected instantaneous rate of return on a \((T - t)\)-year zero-coupon bond is \( \mu_p \).
  - The instantaneous variance is \( \sigma_p^2 \).

\[ ^{\text{aMerton (1970).}} \]

Interest Rate Models (continued)

- Surely \( P(r,T,T) = 1 \) for any \( T \).

- By Ito’s lemma (Theorem 17 on p. 455),
  \[
  dP = \frac{\partial P}{\partial T} \, dT + \frac{\partial P}{\partial r} \, dr + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (dr)^2
  = \left[ -\frac{\partial P}{\partial T} + \mu(r,t) \frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right] dt
  + \sigma(r,t) \frac{\partial P}{\partial r} \, dW.
  \]
Interest Rate Models (concluded)

• Hence,

\[- \frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} = P \mu_p, \hspace{1em} (54)\]

\[\sigma(r, t) \frac{\partial P}{\partial r} = P \sigma_p.\]

• Models with the short rate as the only explanatory variable are called short rate models.

Continuous-Time Derivatives Pricing

I have hardly met a mathematician who was capable of reasoning.
— Plato (428 B.C.–347 B.C.)

Toward the Black-Scholes Differential Equation

• The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.

• The key step is recognizing that the same random process drives both securities.

• As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.

• The removal of uncertainty forces the portfolio's return to be the riskless rate.
**Assumptions**

- The stock price follows $dS = \mu S \, dt + \sigma S \, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at $r$.
- There is unlimited riskless borrowing and lending.
- $t$ is the current time, $T$ is the expiration time, and $\tau \equiv T - t$.

**Black-Scholes Differential Equation**

- Let $C$ be the price of a derivative on $S$.
- From Ito’s lemma (p. 455),
  
  $$dC = \left( \mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) \, dt + \sigma S \frac{\partial C}{\partial S} \, dW.$$  

  - The same $W$ drives both $C$ and $S$.
- Short one derivative and long $\partial C/\partial S$ shares of stock (call it $\Pi$).
- By construction,
  
  $$\Pi = -C + S(\partial C/\partial S).$$

**Black-Scholes Differential Equation (continued)**

- The change in the value of the portfolio at time $dt$ is
  
  $$d\Pi = -dC + \frac{\partial C}{\partial S} \, dS.$$  

- Substitute the formulas for $dC$ and $dS$ into the partial differential equation to yield
  
  $$d\Pi = \left( -\frac{\partial C}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt.$$  

- As this equation does not involve $dW$, the portfolio is riskless during $dt$ time: $d\Pi = r\Pi dt$.

**Black-Scholes Differential Equation (concluded)**

- So
  
  $$\left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt = r \left( C - S \frac{\partial C}{\partial S} \right) dt.$$  

- Equate the terms to finally obtain
  
  $$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$  

- When there is a dividend yield $q$,
  
  $$\frac{\partial C}{\partial t} + (r - q)S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC.$$
Rephrase

• The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,
\[ \Theta + rS\Delta + \frac{1}{2} \sigma^2 S^2 \Gamma = rC. \] (55)

• Identity (55) leads to an alternative way of computing \( \Theta \) numerically from \( \Delta \) and \( \Gamma \).

• When a portfolio is delta-neutral,
\[ \Theta + \frac{1}{2} \sigma^2 S^2 \Gamma = rC. \]

A definite relation thus exists between \( \Gamma \) and \( \Theta \).

PDEs for Asian Options

• Add the new variable \( A(t) \equiv \int_0^t S(u) \, du \).

• Then the value \( V \) of the Asian option satisfies this two-dimensional PDE:\(^a\)
\[ \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial A} = rV. \]

• The terminal conditions are
\[ V(T, S, A) = \max \left( \frac{A}{T} - X, 0 \right) \text{ for call,} \]
\[ V(T, S, A) = \max \left( X - \frac{A}{T}, 0 \right) \text{ for put.} \]

PDEs for Asian Options (continued)

• The two-dimensional PDE produces algorithms similar to that on pp. 316ff.

• But one-dimensional PDEs are available for Asian options.\(^a\)

• For example, Večer (2001) derives the following PDE for Asian calls:
\[ \frac{\partial u}{\partial t} + r \left( 1 - \frac{t}{T} - z \right) \frac{\partial u}{\partial z} + \frac{(1 - \frac{t}{T} - z)^2 \sigma^2}{2} \frac{\partial^2 u}{\partial z^2} = 0 \]
with the terminal condition \( u(T, z) = \max(z, 0) \).

\(^a\)Rogers and Shi (1995), Večer (2001), and Dubois and Lelièvre (2005).

PDEs for Asian Options (concluded)

• For Asian puts:
\[ \frac{\partial u}{\partial t} + r \left( \frac{t}{T} - 1 - z \right) \frac{\partial u}{\partial z} + \frac{(\frac{t}{T} - 1 - z)^2 \sigma^2}{2} \frac{\partial^2 u}{\partial z^2} = 0 \]
with the same terminal condition.

• One-dimensional PDEs lead to highly efficient numerical methods.

\(^a\) Kemna and Vorst (1990).
The payoff implies two ways of looking at the option.

- It is a call on asset 2 with a strike price equal to the future price of asset 1.
- It is a put on asset 1 with a strike price equal to the future value of asset 2.

This is called Margrabe’s formula.
Derivation of Margrabe’s Formula

- Observe first that \( V(x, y, t) \) is homogeneous of degree one in \( x \) and \( y \).
  - That is, \( V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t) \).
  - An exchange option based on \( \lambda \) times the prices of the two assets is thus equal in value to \( \lambda \) original exchange options.
  - Intuitively, this is true because of
    \[
    \max(\lambda S_2(T) - \lambda S_1(T), 0) = \lambda \times \max(S_2(T) - S_1(T), 0)
    \]
    and the perfect market assumption.

Derivation of Margrabe’s Formula (continued)

- The price of asset 2 relative to asset 1 is \( S \equiv S_2/S_1 \).
- The diffusion of \( dS/S \) is \( \sqrt{\sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2} \) by Eq. (52) on p. 463 (proving Eq. (56) on p. 486).
- Hence, the option sells for
  \[
  V(S_1, S_2, t)/S_1 = V(1, S_2/S_1, t)
  \]
  with asset 1 as the numeraire.

Derivation of Margrabe’s Formula (concluded)

- So the Black-Scholes formula applies:
  \[
  \frac{V(S_1, S_2, t)}{S_1} = \frac{V(1, S, t)}{S} = SN(x) - 1 \times e^{-0 \times (T-t)}N(x - \sigma \sqrt{T-t}),
  \]
  where
  \[
  x \equiv \frac{\ln(S/1) + (0 + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = \frac{\ln(S_2/S_1) + (\sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.
  \]
Margrabe’s Formula with Dividends

• Margrabe’s formula is not much more complicated if \( S_i \) pays out a continuous dividend yield of \( q_i \), \( i = 1, 2 \).

• Simply replace each occurrence of \( S_i \) with \( S_i e^{-q_i(T-t)} \) to obtain

\[
V(S_1, S_2, t) = S_2 e^{-q_2(T-t)} N(x) - S_1 e^{-q_1(T-t)} N(x - \sigma \sqrt{T-t}),
\]

where

\[
x = \ln(S_2/S_1) + (q_1 - q_2 + \sigma^2/2)(T-t),
\]

\[
\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2.
\]

Options on Foreign Currencies and Assets

• Correlation options involving foreign currencies and assets can be analyzed either take place in the domestic market or the foreign market before being converted back into the domestic currency.

• In the following, \( S(t) \) denotes the spot exchange rate in terms of the domestic value of one unit of foreign currency.

• We knew from p. 305 that foreign currency is analogous to a stock paying a continuous dividend yield equal to the foreign riskless interest rate \( r_f \) in foreign currency.

Options on Foreign Currencies and Assets (concluded)

• So \( S(t) \) follows the geometric Brownian motion process,

\[
\frac{dS}{S} = (r - r_f) dt + \sigma_s dW_s(t),
\]

in a risk-neutral economy.

• The foreign asset will be assumed to pay a continuous dividend yield of \( q_f \), and its price follows

\[
\frac{dG_f}{G_f} = (\mu_f - q_f) dt + \sigma_f dW_f(t)
\]

in foreign currency.

• \( \rho \) is the correlation between \( dW_s \) and \( dW_f \).

Inverse Exchange Rates

• Suppose we have to work with the inverse of the exchange rate, \( Y \equiv 1/S \), instead of \( S \).

– Because the option payoff is a function of \( Y \); or

– Because the parameters for \( Y \) are quoted in the markets but not \( S \).

• \( Y \) follows

\[
\frac{dY}{Y} = -(r - r_f - \sigma_s^2) dt - \sigma_s dW_s(t)
\]

by Eq. (52) on p. 463.
Inverse Exchange Rates (concluded)

- Hence the volatility of $Y$ equals that of $S$.
  - If a simulation of $S$ gives wildly different sample volatilities for $S$ and $Y$, you probably forgot to take logarithms before calculating the standard deviations.
- The correlation between $Y$ and $G_f$ equals

\[
\rho = \frac{E[(G_f - X_f \times e^{-r_f \tau})N(-x + \sigma_f \sqrt{\tau})] - E[G_f e^{-q_f \tau} N(-x)]}{\sqrt{E[G_f^2 e^{-2q_f \tau} N(-x)]}}
\]

as the correlation between $S$ and $G_f$ is $\rho$.

Foreign Equity Options

- From Eq. (26) on p. 255, a European option on the foreign asset $G_f$ with the terminal payoff $S(T) \times \max(G_f(T) - X_f, 0)$ is worth

\[
P_t = X_f e^{-r_f \tau} N(-x + \sigma_f \sqrt{\tau}) - G_f e^{-q_f \tau} N(-x)
\]
in foreign currency.
- They will fetch $SC_f$ and $SP_f$, respectively, in domestic currency.
- These options are called foreign equity options struck in foreign currency.

Foreign Domestic Options

- Foreign equity options fundamentally involve values in the foreign currency.
- A foreign equity call may allow the holder to participate in a foreign market rally.
- But the profits can be wiped out if the foreign currency depreciates against the domestic currency.
- What is really needed is a call in domestic currency with a payoff of $\max(S(T) G_f(T) - X, 0)$.
  - For foreign equity options, the strike price in domestic currency is the uncertain $S(T) X_f$.
  - This is called a foreign domestic option.
Pricing of Foreign Domestic Options

- To foreign investors, this call is an option to exchange $X$ units of domestic currency (foreign currency to them) for one share of foreign asset (domestic asset to them).
- It is an exchange option, that is.
- By Eq. (57) on p. 491, its price in foreign currency equals

$$G_t e^{-q f \tau} N(x) - \frac{X}{S} e^{-r \tau} N(x - \sigma \sqrt{\tau}),$$

$$x \equiv \ln(G_t S/X) + (r - q f + \sigma^2/2) \tau \sigma \sqrt{\tau},$$

$$\sigma^2 \equiv \sigma^2_a + 2 \rho \sigma_a \sigma_f + \sigma^2_f.$$

Cross-Currency Options

- A cross-currency option is an option in which the currency of the strike price is different from the currency in which the underlying asset is denominated.
  - An option to buy 100 yen at a strike price of 1.18 Canadian dollars provides one example.
- Usually, a third currency, the U.S. dollar, is involved because of the lack of relevant exchange-traded options for the two currencies in question (yen and Canadian dollars in the above example).
- So the notations below will be slightly different.

Pricing of Foreign Domestic Options (concluded)

- The domestic price is therefore

$$C = SG_t e^{-q f \tau} N(x) - X e^{-r \tau} N(x - \sigma \sqrt{\tau}).$$

- Similarly, a put has a price of

$$P = X e^{-r \tau} N(-x + \sigma \sqrt{\tau}) - SG_t e^{-q f \tau} N(-x).$$

Cross-Currency Options (continued)

- Let $S_A$ denote the price of the foreign asset and $S_C$ the price of currency C that the strike price $X$ is based on.
- Both $S_A$ and $S_C$ are in U.S. dollars, say.
- If $S$ is the price of the foreign asset as measured in currency C, then we have the triangular arbitrage

$$S = S_A / S_C.$$

*Triangular arbitrage* had been known for centuries. See Montesquieu’s *The Spirit of Laws.*
Cross-Currency Options (concluded)

- Assume $S_A$ and $S_C$ follow the geometric Brownian motion processes $dS_A/S_A = \mu_A dt + \sigma_A dW_A$ and $dS_C/S_C = \mu_C dt + \sigma_C dW_C$, respectively.
  - Parameters $\sigma_A$, $\sigma_C$, and $\rho$ can be inferred from exchange-traded options.
- By an exercise in the text,
  $$dS/S = (\mu_A - \mu_C + \sigma_C^2 - \rho \sigma_A \sigma_C) dt + \sigma_A dW_A - \sigma_C dW_C,$$
  where $\rho$ is the correlation between $dW_A$ and $dW_C$.
- The volatility of $dS/S$ is hence $(\sigma_A^2 - 2 \rho \sigma_A \sigma_C + \sigma_C^2)^{1/2}$.

Quanto Options (continued)

- The process $U \equiv \hat{S} G_t$ in a risk-neutral economy follows
  $$\frac{dU}{U} = (r_t - q_t - \rho \sigma_s \sigma_l) dt + \sigma_l dW$$
  in domestic currency.
- Hence, it can be treated as a stock paying a continuous dividend yield of $q \equiv r - r_t + q_t + \rho \sigma_s \sigma_l$.
- Apply Eq. (26) on p. 255 to obtain
  $$C = \hat{S}(G_t e^{-q t} N(x) - X_t e^{-r t} N(x - \sigma_l \sqrt{t}))$$
  $$P = \hat{S}(X_t e^{-r t} N(-x + \sigma_l \sqrt{t}) - G_t e^{-q t} N(-x))$$
  where $x \equiv \frac{\ln(G_t/X_t) + (r - q + \sigma_l^2/2) t}{\sigma_l \sqrt{t}}$.

Quanto Options

- Consider a call with a terminal payoff
  $$\hat{S} \times \max(G_t(T) - X_t, 0)$$
  in domestic currency, where $\hat{S}$ is a constant.
- This amounts to fixing the exchange rate to $\hat{S}$.
  - For instance, a call on the Nikkei 225 futures, if it existed, fits this framework with $\hat{S} = 5$ and $G_t$ denoting the futures price.
- A guaranteed exchange rate option is called a quanto option or simply a quanto.

Quanto Options (concluded)

- In general, a quanto derivative has nominal payments in the foreign currency which are converted into the domestic currency at a fixed exchange rate.
- A cross-rate swap, for example, is like a currency swap except that the foreign currency payments are converted into the domestic currency at a fixed exchange rate.
- Quanto derivatives form a rapidly growing segment of international financial markets.