Example

• Consider the stochastic process

\[ \{ Z_n = \sum_{i=1}^{n} X_i, n \geq 1 \}, \]

where \( X_i \) are independent random variables with zero mean.

• This process is a martingale because

\[
E[ Z_{n+1} | Z_1, Z_2, \ldots, Z_n ] = E[ Z_n + X_{n+1} | Z_1, Z_2, \ldots, Z_n ] = E[ Z_n | Z_1, Z_2, \ldots, Z_n ] + E[ X_{n+1} | Z_1, Z_2, \ldots, Z_n ] = Z_n + E[ X_{n+1} ] = Z_n.
\]

Probability Measure

• A martingale is defined with respect to a probability measure, under which the expectation is taken.
  – A probability measure assigns probabilities to states of the world.

• A martingale is also defined with respect to an information set.
  – In the characterizations (41)–(42) on p. 397, the information set contains the current and past values of \( X \) by default.
  – But it needs not be so.

Probability Measure (continued)

• A stochastic process \( \{ X(t), t \geq 0 \} \) is a martingale with respect to information sets \( \{ I_t \} \) if, for all \( t \geq 0 \),

\[
E[ | X(t) | ] < \infty \quad \text{and} \quad E[ X(u) | I_t ] = X(t)
\]

for all \( u > t \).

• The discrete-time version: For all \( n > 0 \),

\[
E[ X_{n+1} | I_n ] = X_n,
\]

given the information sets \( \{ I_n \} \).
Example

- Consider the stochastic process \( \{ Z_n - n\mu, n \geq 1 \} \).
  - \( Z_n \equiv \sum_{i=1}^{n} X_i \).
  - \( X_1, X_2, \ldots \) are independent random variables with mean \( \mu \).
- Now,
  \[
  E[ Z_{n+1} - (n + 1) \mu \mid X_1, X_2, \ldots, X_n ] = E[ Z_{n+1} \mid X_1, X_2, \ldots, X_n ] - (n + 1) \mu \\
  = Z_n + \mu - (n + 1) \mu \\
  = Z_n - n\mu.
  \]

Martingale Pricing

- Recall that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy.
- This principle can be generalized using the concept of martingale.
- Recall the recursive valuation of European option via
  \[
  C = \left[ pC_u + (1-p)C_d \right]/R.
  \]
  - \( p \) is the risk-neutral probability.
  - \$1 grows to \$\( R \) in a period.

Example (concluded)

- Define
  \[
  I_n \equiv \{ X_1, X_2, \ldots, X_n \}.
  \]
- Then
  \[
  \{ Z_n - n\mu, n \geq 1 \}
  \]
  is a martingale with respect to \( \{ I_n \} \).

Martingale Pricing (continued)

- Let \( C(i) \) denote the value of the option at time \( i \).
- Consider the discount process
  \[
  \{ C(i)/R^i, i = 0, 1, \ldots, n \}.
  \]
- Then,
  \[
  E \left[ \frac{C(i + 1)}{R^{i+1}} \mid C(i) = C \right] = \frac{pC_u + (1-p)C_d}{R^{i+1}} = \frac{C}{R^i}.
  \]
Martingale Pricing (continued)

- In general,
  \[
  E \left[ \frac{C(k)}{R^k} \bigg| C(i) = C \right] = \frac{C}{R^i}, \quad i \leq k. \quad (44)
  \]

- The discount process is a martingale:
  \[
  \frac{C(i)}{R^i} = E^\pi_i \left[ \frac{C(k)}{R^k} \right], \quad i \leq k. \quad (45)
  \]
  - \( E^\pi_i \) is taken under the risk-neutral probability conditional on the price information up to time \( i \).
  - This risk-neutral probability is also called the EMM, or the equivalent martingale (probability) measure.

- If interest rates are stochastic, then \( M(j) \) is a random variable.
  - \( M(0) = 1 \).
  - \( M(j) \) is known at time \( j - 1 \).
- Identity (46) on p. 411 is the general formulation of risk-neutral valuation.

**Theorem 14** A discrete-time model is arbitrage-free if and only if there exists a probability measure such that the discount process is a martingale. This probability measure is called the risk-neutral probability measure.

Martingale Pricing (continued)

- In general, Eq. (45) holds for all assets, not just options.
- When interest rates are stochastic, the equation becomes
  \[
  \frac{C(i)}{M(i)} = E^\pi_i \left[ \frac{C(k)}{M(k)} \right], \quad i \leq k. \quad (46)
  \]
  - \( M(j) \) is the balance in the money market account at time \( j \) using the rollover strategy with an initial investment of $1.
  - So it is called the bank account process.
- It says the discount process is a martingale under \( \pi \).

Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is
    \[
    p_t F u + (1 - p_t) F d = F \left( \frac{1 - d}{u - d} u + \frac{u - 1}{u - d} d \right) = F
    \]
    (p. 374).
  - Can be generalized to
    \[
    F_i = E^\pi_i \left[ F_k \right], \quad i \leq k,
    \]
    where \( F_i \) is the futures price at time \( i \).
  - It holds under stochastic interest rates.
Martingale Pricing and Numeraire

- The martingale pricing formula (46) on p. 411 uses the money market account as numeraire.\(^a\)
  - It expresses the price of any asset \textit{relative to} the money market account.
- The money market account is not the only choice for numeraire.
- Suppose asset \(S\)'s value is positive at all times.

\(^a\)Leon Walras (1834–1910).

Example

- Take the binomial model with two assets.
- In a period, asset one’s price can go from \(S\) to \(S_1\) or \(S_2\).
- In a period, asset two’s price can go from \(P\) to \(P_1\) or \(P_2\).
- Assume
  \[
  \frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2}
  \]
  to rule out arbitrage opportunities.

Martingale Pricing and Numeraire (concluded)

- Choose \(S\) as numeraire.
- Martingale pricing says there exists a risk-neutral probability \(\pi\) under which the relative price of any asset \(C\) is a martingale:
  \[
  \frac{C(i)}{S(i)} = E^\pi_i \left[ \frac{C(k)}{S(k)} \right], \quad i \leq k.
  \]
  - \(S(j)\) denotes the price of \(S\) at time \(j\).
- So the discount process remains a martingale.

Example (continued)

- For any derivative security, let \(C_1\) be its price at time one if asset one’s price moves to \(S_1\).
- Let \(C_2\) be its price at time one if asset one’s price moves to \(S_2\).
- Replicate the derivative by solving
  \[
  \alpha S_1 + \beta P_1 = C_1, \\
  \alpha S_2 + \beta P_2 = C_2,
  \]
  using \(\alpha\) units of asset one and \(\beta\) units of asset two.
Example (continued)

- This yields
  \[ \alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2} \]

- The derivative costs
  \[ C = \alpha S + \beta P = \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{PS_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2. \]

Example (concluded)

- It is easy to verify that
  \[ \frac{C}{P} = p \frac{C_1}{P_1} + (1-p) \frac{C_2}{P_2}. \]
  Above,
  \[ p = \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}. \]

- The derivative’s price using asset two as numeraire is thus a martingale under the risk-neutral probability \( p \).

- The expected returns of the two assets are irrelevant.

Brownian Motion

- Brownian motion is a stochastic process \( \{ X(t), t \geq 0 \} \) with the following properties.
  1. \( X(0) = 0 \), unless stated otherwise.
  2. for any \( 0 \leq t_0 < t_1 < \cdots < t_n \), the random variables \( X(t_k) - X(t_{k-1}) \) for \( 1 \leq k \leq n \) are independent.
  3. for \( 0 \leq s < t \), \( X(t) - X(s) \) is normally distributed with mean \( \mu(t-s) \) and variance \( \sigma^2(t-s) \), where \( \mu \) and \( \sigma \neq 0 \) are real numbers.

Brownian Motion (concluded)

- Such a process will be called a \( (\mu, \sigma) \) Brownian motion with drift \( \mu \) and variance \( \sigma^2 \).
- The existence and uniqueness of such a process is guaranteed by Wiener’s theorem.
- Although Brownian motion is a continuous function of \( t \) with probability one, it is almost nowhere differentiable.
- The \( (0, 1) \) Brownian motion is also called the Wiener process.

---

\( ^a \)Robert Brown (1773–1858).
\( ^b \)So \( X(t) - X(s) \) is independent of \( X(r) \) for \( r \leq s < t \).
**Example**

- If \( \{ X(t), t \geq 0 \} \) is the Wiener process, then
  \[ X(t) - X(s) \sim N(0, t - s). \]
- A \((\mu, \sigma)\) Brownian motion \( Y = \{ Y(t), t \geq 0 \} \) can be expressed in terms of the Wiener process:
  \[ Y(t) = \mu t + \sigma X(t). \]
- As \( Y(t + s) - Y(t) \sim N(\mu s, \sigma^2 s) \), uncertainty about the future value of \( Y \) grows as the square root of how far we look into the future.

**Brownian Motion as Limit of Random Walk**

**Claim 1** A \((\mu, \sigma)\) Brownian motion is the limiting case of random walk.

- A particle moves \( \Delta x \) to the left with probability \( 1 - p \).
- It moves to the right with probability \( p \) after \( \Delta t \) time.
- Assume \( n \equiv t/\Delta t \) is an integer.
- Its position at time \( t \) is
  \[ Y(t) \equiv \Delta x (X_1 + X_2 + \cdots + X_n). \]

**Brownian Motion as Limit of Random Walk**

(continued)

- Therefore,
  \[
  \begin{align*}
  E[Y(t)] &= n(\Delta x)(2p - 1), \\
  \text{Var}[Y(t)] &= n(\Delta x)^2 \left[ 1 - (2p - 1)^2 \right].
  \end{align*}
  \]
- With \( \Delta x \equiv \sigma \sqrt{\Delta t} \) and \( p \equiv \left[ 1 + (\mu/\sigma) \sqrt{\Delta t} \right] / 2 \),
  \[
  \begin{align*}
  E[Y(t)] &= n \sigma \sqrt{\Delta t} (\mu/\sigma) \sqrt{\Delta t} = \mu t, \\
  \text{Var}[Y(t)] &= n \sigma^2 \Delta t \left[ 1 - (\mu/\sigma)^2 \Delta t \right] \to \sigma^2 t,
  \end{align*}
  \]
  as \( \Delta t \to 0 \).
Brownian Motion as Limit of Random Walk (concluded)

• Thus, \( \{ Y(t), t \geq 0 \} \) converges to a \((\mu, \sigma)\) Brownian motion by the central limit theorem.

• Brownian motion with zero drift is the limiting case of symmetric random walk by choosing \( \mu = 0 \).

• Note that
\[
\text{Var}[Y(t + \Delta t) - Y(t)] = \text{Var}[\Delta x X_{n+1}] = (\Delta x)^2 \times \text{Var}[X_{n+1}] = \sigma^2 \Delta t.
\]

• Similarity to the the BOPM: The \( p \) is identical to the probability in Eq. (25) on p. 234 and \( \Delta x = \ln u \).

Geometric Brownian Motion

• Let \( X \equiv \{ X(t), t \geq 0 \} \) be a Brownian motion process.

• The process \( \{ Y(t) \equiv e^{X(t)}, t \geq 0 \} \),

is called geometric Brownian motion.

• Suppose further that \( X \) is a \((\mu, \sigma)\) Brownian motion.

• \( X(t) \sim N(\mu t, \sigma^2 t) \) with moment generating function
\[
E[e^{sX(t)}] = E[Y(t)^s] = e^{\mu ts + (\sigma^2 ts^2/2)}
\]
from Eq. (17) on p 138.

Geometric Brownian Motion (continued)

• In particular,
\[
E[Y(t)] = e^{\mu t + (\sigma^2 t/2)},
\]
\[
\text{Var}[Y(t)] = E[Y(t)^2] - E[Y(t)]^2 = e^{2\mu t + \sigma^2 t} - \left(e^{\sigma^2 t} - 1\right).
\]
Geometric Brownian Motion (continued)

- It is useful for situations in which percentage changes are independent and identically distributed.
- Let $Y_n$ denote the stock price at time $n$ and $Y_0 = 1$.
- Assume relative returns
  \[ X_i = \frac{Y_i}{Y_{i-1}} \]
  are independent and identically distributed.

Geometric Brownian Motion (concluded)

- Then
  \[ \ln Y_n = \sum_{i=1}^{n} \ln X_i \]
  is a sum of independent, identically distributed random variables.
- Thus $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion.
  - And $\{ Y_n, n \geq 0 \}$ is approximately geometric Brownian motion.

Continuous-Time Financial Mathematics

A proof is that which convinces a reasonable man; a rigorous proof is that which convinces an unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a divine madness of the human spirit.
— Alfred North Whitehead (1861–1947),
    Science and the Modern World
Stochastic Integrals

- Use $W \equiv \{ W(t), t \geq 0 \}$ to denote the Wiener process.
- The goal is to develop integrals of $X$ from a class of stochastic processes,

$$I_t(X) \equiv \int_0^t X \, dW, \quad t \geq 0.$$  

- $I_t(X)$ is a random variable called the stochastic integral of $X$ with respect to $W$.
- The stochastic process $\{ I_t(X), t \geq 0 \}$ will be denoted by $\int X \, dW$.

\footnote{Ito (1915–).}

Ito Integral

- A theory of stochastic integration.
- As with calculus, it starts with step functions.
- A stochastic process $\{ X(t) \}$ is simple if there exist $0 = t_0 < t_1 < \cdots$ such that

$$X(t) = X(t_{k-1}) \text{ for } t \in [t_{k-1}, t_k), \; k = 1, 2, \ldots$$

for any realization (see figure next page).

Stochastic Integrals (concluded)

- Typical requirements for $X$ in financial applications are:
  - $\text{Prob} \left[ \int_0^t X^2(s) \, ds < \infty \right] = 1$ for all $t \geq 0$ or the stronger $\int_0^t \mathbb{E}[X^2(s)] \, ds < \infty$.
  - The information set at time $t$ includes the history of $X$ and $W$ up to that point in time.
  - But it contains nothing about the evolution of $X$ or $W$ after $t$ (nonanticipating, so to speak).
  - The future cannot influence the present.
- $\{ X(s), 0 \leq s \leq t \}$ is independent of $\{ W(t + u) - W(t), u > 0 \}$. 
Ito Integral (continued)

- The Ito integral of a simple process is defined as
  \[
  \mathcal{I}_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)[W(t_{k+1}) - W(t_k)],
  \]
  where \( t_n = t \).
  - The integrand \( X \) is evaluated at \( t_k \), not \( t_{k+1} \).
- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.

Ito Integral (concluded)

- It is a fundamental fact that \( \int X \, dW \) is continuous almost surely.
- The following theorem says the Ito integral is a martingale.
- A corollary is the mean value formula
  \[
  E \left[ \int_a^b X \, dW \right] = 0.
  \]

**Theorem 15** The Ito integral \( \int X \, dW \) is a martingale.

Discrete Approximation

- Let \( X = \{ X(t), t \geq 0 \} \) be a general stochastic process.
- Then there exists a random variable \( \mathcal{I}_t(X) \), unique almost certainly, such that \( \mathcal{I}_t(X_n) \) converges in probability to \( \mathcal{I}_t(X) \) for each sequence of simple stochastic processes \( X_1, X_2, \ldots \) such that \( X_n \) converges in probability to \( X \).
- If \( X \) is continuous with probability one, then \( \mathcal{I}_t(X_n) \) converges in probability to \( \mathcal{I}_t(X) \) as \( \delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1}) \) goes to zero.

- Recall Eq. (47) on p. 438.
- The following simple stochastic process \( \{ \tilde{X}(t) \} \) can be used in place of \( X \) to approximate the stochastic integral \( \int_0^b X \, dW \),
  \[
  \tilde{X}(s) \equiv X(t_{k-1}) \text{ for } s \in [t_{k-1}, t_k), k = 1, 2, \ldots, n.
  \]
- Note the nonanticipating feature of \( \tilde{X} \).
  - The information up to time \( s \),
    \[
    \{ \tilde{X}(t), W(t), 0 \leq t \leq s \},
    \]
    cannot determine the future evolution of \( X \) or \( W \).
Discrete Approximation (concluded)

• Suppose we defined the stochastic integral as

\[ \sum_{k=0}^{n-1} X(t_{k+1})[W(t_{k+1}) - W(t_k)]. \]

• Then we would be using the following different simple stochastic process in the approximation,

\[ \hat{Y}(s) \equiv X(t_k) \text{ for } s \in [t_{k-1}, t_k), k = 1, 2, \ldots , n. \]

• This clearly anticipates the future evolution of \( X \).

Ito Process

• The stochastic process \( X = \{ X_t, t \geq 0 \} \) that solves

\[ X_t = X_0 + \int_0^t a(X_s, s) \, ds + \int_0^t b(X_s, s) \, dW_s, \quad t \geq 0 \]

is called an Ito process.

– \( X_0 \) is a scalar starting point.

– \( \{ a(X_t, t) : t \geq 0 \} \) and \( \{ b(X_t, t) : t \geq 0 \} \) are stochastic processes satisfying certain regularity conditions.

• The terms \( a(X_t, t) \) and \( b(X_t, t) \) are the drift and the diffusion, respectively.

Ito Process (continued)

• A shorthand\(^a\) is the following stochastic differential equation for the Ito differential \( dX_t \),

\[ dX_t = a(X_t, t) \, dt + b(X_t, t) \, dW_t. \quad (48) \]

– Or simply \( dX_t = a_t \, dt + b_t \, dW_t \).

• This is Brownian motion with an instantaneous drift \( a_t \) and an instantaneous variance \( b_t^2 \).

• \( X \) is a martingale if the drift \( a_t \) is zero by Theorem 15 (p. 440).

\(^a\)Paul Langevin (1904).
Ito Process (concluded)

- $dW$ is normally distributed with mean zero and variance $dt$.
- An equivalent form to Eq. (48) is
  \[ dX_t = a_t \, dt + b_t \sqrt{dt} \, \xi, \]
  \[ \xi \sim N(0, 1). \]
- This formulation makes it easy to derive Monte Carlo simulation algorithms.

Euler Approximation

- The following approximation follows from Eq. (49),
  \[ \hat{X}(t_{n+1}) = \tilde{X}(t_n) + a(\tilde{X}(t_n), t_n) \Delta t + b(\tilde{X}(t_n), t_n) \Delta W(t_n), \]
  \[ (50) \]
  where $t_n \equiv n \Delta t$.
- It is called the Euler or Euler-Maruyama method.
- Under mild conditions, $\tilde{X}(t_n)$ converges to $X(t_n)$.
- Recall that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$ instead of $W(t_n) - W(t_{n-1})$.

More Discrete Approximations

- Under fairly loose regularity conditions, approximation (50) on p. 447 can be replaced by
  \[ \hat{X}(t_{n+1}) = \tilde{X}(t_n) + a(\tilde{X}(t_n), t_n) \Delta t + b(\tilde{X}(t_n), t_n) \sqrt{\Delta t} \, Y(t_n). \]
  - $Y(t_0), Y(t_1), \ldots$ are independent and identically distributed with zero mean and unit variance.

More Discrete Approximations (concluded)

- A simpler discrete approximation scheme:
  \[ \tilde{X}(t_{n+1}) = \tilde{X}(t_n) + a(\tilde{X}(t_n), t_n) \Delta t + b(\tilde{X}(t_n), t_n) \sqrt{\Delta t} \, \xi. \]
  - Prob[$\xi = 1$] = Prob[$\xi = -1$] = 1/2.
  - Note that $E[\xi] = 0$ and Var[$\xi$] = 1.
- This clearly defines a binomial model.
- As $\Delta t$ goes to zero, $\tilde{X}$ converges to $X$. 
Trading and the Ito Integral

- Consider an Ito process \( dS_t = \mu_t dt + \sigma_t dW_t \).
  - \( S_t \) is the vector of security prices at time \( t \).
- Let \( \phi_t \) be a trading strategy denoting the quantity of each type of security held at time \( t \).
- Hence the stochastic process \( \phi_t S_t \) is the value of the portfolio \( \phi_t \) at time \( t \).
- \( \phi_t dS_t \equiv \phi_t (\mu_t dt + \sigma_t dW_t) \) represents the change in the value from security price changes occurring at time \( t \).

Ito’s Lemma

A smooth function of an Ito process is itself an Ito process.

**Theorem 16** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is twice continuously differentiable and \( dX = a_t dt + b_t dW \). Then \( f(X) \) is the Ito process,

\[
\begin{align*}
  f(X_t) &= f(X_0) + \int_0^t f'(X_s) a_s ds + \int_0^t f'(X_s) b_s dW_s \\
         &+ \frac{1}{2} \int_0^t f''(X_s) b_s^2 ds
\end{align*}
\]

for \( t \geq 0 \).

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,
  \[
  G_T(\phi) \equiv \int_0^T \phi_t dS_t = \int_0^T \phi_t \mu_t dt + \int_0^T \phi_t \sigma_t dW_t,
  \]
measures the gains realized by the trading strategy over the period \([0, T]\).

Ito’s Lemma (continued)

- In differential form, Ito’s lemma becomes
  \[
  df(X) = f'(X) a dt + f'(X) b dW + \frac{1}{2} f''(X) b^2 dt.
  \tag{51}
  \]
- Compared with calculus, the interesting part is the third term on the right-hand side.
- A convenient formulation of Ito’s lemma is
  \[
  df(X) = f'(X) dX + \frac{1}{2} f''(X) (dX)^2.
  \]
Ito’s Lemma (continued)

- We are supposed to multiply out
  $(dX)^2 = (a \, dt + b \, dW)^2$ symbolically according to

<table>
<thead>
<tr>
<th>$\times$</th>
<th>$dW$</th>
<th>$dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dW$</td>
<td>$dt$</td>
<td>0</td>
</tr>
<tr>
<td>$dt$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- The $(dW)^2 = dt$ entry is justified by a known result.
- This form is easy to remember because of its similarity to the Taylor expansion.

Theorem 17 (Higher-Dimensional Ito’s Lemma) Let $W_1, W_2, \ldots, W_n$ be independent Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable and $X_i$ is an Ito process with $dX_i = a_i \, dt + \sum_{j=1}^n b_{ij} \, dW_j$. Then $df(X)$ is an Ito process with the differential,

$$df(X) = \sum_{i=1}^m f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m f_{ik}(X) \, dX_i \, dX_k,$$

where $f_i \equiv \partial f / \partial x_i$ and $f_{ik} \equiv \partial^2 f / \partial x_i \partial x_k$.

Ito’s Lemma (continued)

- The multiplication table for Theorem 17 is

<table>
<thead>
<tr>
<th>$\times$</th>
<th>$dW_k$</th>
<th>$\delta_{ik} , dt$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dW_k$</td>
<td>$\delta_{ik}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$dt$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 18 (Alternative Ito’s Lemma) Let $W_1, W_2, \ldots, W_m$ be Wiener processes and $X \equiv (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable and $X_i$ is an Ito process with $dX_i = a_i \, dt + b_i \, dW_i$. Then $df(X)$ is the following Ito process,

$$df(X) = \sum_{i=1}^m f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m f_{ik}(X) \, dX_i \, dX_k.$$
Ito’s Lemma (concluded)

- The multiplication table for Theorem 18 is

<table>
<thead>
<tr>
<th>×</th>
<th>dW_i</th>
<th>dW_k</th>
<th>dW_i</th>
<th>dW_k</th>
<th>dt</th>
</tr>
</thead>
<tbody>
<tr>
<td>dW_k</td>
<td>ρ_{ik} dt</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dt</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Here, ρ_{ik} denotes the correlation between dW_i and dW_k.