Delivery and Hedging

- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.
- Changes in futures prices usually track those in spot prices.
- This makes hedging possible.
- Before the delivery date, the futures price could be above or below the spot price.

Daily Cash Flows

- Let F_i denote the futures price at the end of day i.
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$\begin{aligned}&(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1})\\ &= F_n - F_0 = S_T - F_0.\end{aligned}$$

- A futures contract has the same accumulated payoff $S_T - F_0$ as a forward contract.
- The actual payoff may differ because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Forward and Futures Prices[a]

Futures price equals forward price if interest rates are nonstochastic![b]

- See text for proof.

[a] Cox, Ingersoll, and Ross (1981).

[b] This "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
  - A call holder acquires a long futures position.
  - A put holder acquires a short futures position.
- The futures contract is then marked to market, and the futures position of the two parties will be at the prevailing futures price.

Futures Options (concluded)

- It works as if the call writer delivered a futures contract to the option holder and paid the holder the prevailing futures price minus the strike price.
- It works as if the put writer took delivery a futures contract from the option holder and paid the holder the strike price minus the prevailing futures price.
- The amount of money that changes hands upon exercise is the difference between the strike price and the prevailing futures price.

Forward Options

- Similar to futures options except that what is delivered is a forward contract with a delivery price equal to the option's strike price.
  - Exercising a call forward option results in a long position in a forward contract.
  - Exercising a put forward option results in a short position in a forward contract.
- Exercising a forward option incurs no immediate cash flows.

Example

- Consider a call with strike $100 and an expiration date in September.
- The underlying asset is a forward contract with a delivery date in December.
- Suppose the forward price in July is $110.
- Upon exercise, the call holder receives a forward contract with a delivery price of $100.
- If an offsetting position is then taken in the forward market, a $10 profit in December will be assured.
- A call on the futures would realize the $10 profit in July.

Some Pricing Relations

- Let delivery take place at time T, the current time be 0, and the option on the futures or forward contract have expiration date t $(t \leq T)$.
- Assume a constant, positive interest rate.
- Although forward price equals futures price, a forward option does not have the same value as a futures option.
- The payoffs at time t are

$$\begin{aligned} \text{futures option} &= \max(F_t - X, 0), & (37)\\ \text{forward option} &= \max(F_t - X, 0)\, e^{-r(T-t)}. & (38) \end{aligned}$$

Some Pricing Relations (concluded)

- A European futures option is worth the same as the corresponding European option on the underlying asset if the futures contract has the same maturity as the options.
  - Futures price equals spot price at maturity.
  - This conclusion is independent of the model for the spot price.

Put-Call Parity

The put-call parity is slightly different from the one in Eq. (19) on p. 177.

Theorem 11 *(1) For European options on futures contracts,* $C = P - (X - F)\, e^{-rt}$. *(2) For European options on forward contracts,* $C = P - (X - F)\, e^{-rT}$.

- See text for proof.

Early Exercise and Forward Options

The early exercise feature is not valuable.

Theorem 12 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

- See text for proof.

Early exercise may be optimal for American futures options even if the underlying asset generates no payouts.

Theorem 13 *American futures options may be exercised optimally before expiration.*

Black Model[a]

- Formulas for European futures options:

$$\begin{aligned} C &= Fe^{-rt}N(x) - Xe^{-rt}N(x - \sigma\sqrt{t}), \qquad (39) \\ P &= Xe^{-rt}N(-x + \sigma\sqrt{t}) - Fe^{-rt}N(-x), \end{aligned}$$

  where $x \equiv \frac{\ln(F/X) + (\sigma^2/2)\,t}{\sigma\sqrt{t}}$.
- Formulas (39) are related to those for options on a stock paying a continuous dividend yield.
- In fact, they are exactly Eqs. (26) on p. 255 with the dividend yield q set to the interest rate r and the stock price S replaced by the futures price F.

[a] Black (1976).

Black Model (concluded)

- This observation incidentally proves Theorem 13 (p. 371).
- For European forward options, just multiply the above formulas by $e^{-r(T-t)}$.
  - Forward options differ from futures options by a factor of $e^{-r(T-t)}$ based on Eqs. (37)–(38).

Binomial Model for Forward and Futures Options

- Futures price behaves like a stock paying a continuous dividend yield of r.
- Under the BOPM, the risk-neutral probability for the futures price is

$$p_{\mathrm{f}} \equiv (1-d)/(u-d)$$

  by Eq. (27) on p. 256.
  - The futures price moves from F to Fu with probability p_{f} and to Fd with probability $1 - p_{\mathrm{f}}$.
- The binomial tree algorithm for forward options is identical except that Eq. (38) on p. 368 is the payoff.

Spot and Futures Prices under BOPM

- The futures price is related to the spot price via $F = Se^{rT}$ if the underlying asset pays no dividends.
- The stock price moves from $S = Fe^{-rT}$ to

$$Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$$

  with probability p_{f} per period.
- The stock price moves from $S = Fe^{-rT}$ to

$$Sde^{r\Delta t}$$

  with probability $1 - p_{\mathrm{f}}$ per period.

Negative Probabilities Revisited

- As $0 < p_\mathrm{f} < 1$, we have $0 < 1 - p_\mathrm{f} < 1$ as well.
- The problem of negative risk-neutral probabilities is now solved:
  - Suppose the stock pays a continuous dividend yield of q.
  - Build the tree for the futures price F of the futures contract expiring at the same time as the option.
  - Calculate S from F at each node via $S = Fe^{-(r-q)(T-t)}$.

Swaps

- Swaps are agreements between two counterparties to exchange cash flows in the future according to a predetermined formula.
- There are two basic types of swaps: interest rate and currency.
- An interest rate swap occurs when two parties exchange interest payments periodically.
- Currency swaps are agreements to deliver one currency against another (our focus here).

Currency Swaps

- A currency swap involves two parties to exchange cash flows in different currencies.
- Consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	Yen
A	$D_\mathrm{A}\%$	$Y_\mathrm{A}\%$
B	$D_\mathrm{B}\%$	$Y_\mathrm{B}\%$

- Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars.

Currency Swaps (continued)

- A straightforward scenario is for A to borrow yen at $Y_\mathrm{A}\%$ and B to borrow dollars at $D_\mathrm{B}\%$.
- But suppose A is *relatively* more competitive in the dollar market than the yen market, and vice versa for B.
  - That is, $Y_\mathrm{B} - Y_\mathrm{A} < D_\mathrm{B} - D_\mathrm{A}$.
- Consider this alternative arrangement:
  - A borrows dollars.
  - B borrows yen.
  - They enter into a currency swap with a bank as the intermediary.

Currency Swaps (concluded)

- The counterparties exchange principal at the beginning and the end of the life of the swap.
- This act transforms A's loan into a yen loan and B's yen loan into a dollar loan.
- The total gain is $((D_{\mathrm{B}} - D_{\mathrm{A}}) - (Y_{\mathrm{B}} - Y_{\mathrm{A}}))\%$:
  - The total interest rate is originally $(Y_{\mathrm{A}} + D_{\mathrm{B}})\%$.
  - The new arrangement has a smaller total rate of $(D_{\mathrm{A}} + Y_{\mathrm{B}})\%$.
- Transactions will happen only if the gain is distributed so that the cost to each party is less than the original.

Example

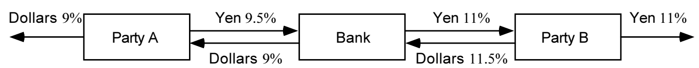
- A and B face the following borrowing rates:

	Dollars	Yen
A	9%	10%
B	12%	11%

- A wants to borrow yen, and B wants to borrow dollars.
- A can borrow yen directly at 10%.
- B can borrow dollars directly at 12%.

Example (concluded)

- As the rate differential in dollars (3%) is different from that in yen (1%), a currency swap with a total saving of $3 - 1 = 2\%$ is possible.
- A is relatively more competitive in the dollar market, and B the yen market.
- Figure next page shows an arrangement which is beneficial to all parties involved.
  - A effectively borrows yen at 9.5%. B borrows dollars at 11.5%.
  - The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.



As a Package of Cash Market Instruments

- Assume no default risk.
- Take B on p. 383 as an example.
- The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest.
- The pricing formula is $SP_{\mathrm{Y}} - P_{\mathrm{D}}$.
  - P_{D} is the dollar bond's value in dollars.
  - P_{Y} is the yen bond's value in yen.
  - S is the \$/yen spot exchange rate.

As a Package of Cash Market Instruments (concluded)

- The value of a currency swap depends on the term structures of interest rates in the currencies involved and the spot exchange rate.
- It has zero value when $SP_{\mathrm{Y}} = P_{\mathrm{D}}$.

Example

- Take a two-year swap on p. 383 with principal amounts of US\$1 million and 100 million yen.
- The payments are made once a year.
- The spot exchange rate is 90 yen/\$ and the term structures are flat in both nations—8% in the U.S. and 9% in Japan.
- For B, the value of the swap is (in millions of USD)

$$\begin{aligned}&\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09\times 2} + 111 \times e^{-0.09\times 3}\right)\\ -&\left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08\times 2} + 1.115 \times e^{-0.08\times 3}\right) \quad = \quad 0.074.\end{aligned}$$

As a Package of Forward Contracts

- From Eq. (35) on p. 354, the forward contract maturing i years from now has a dollar value of

$$f_i \equiv (SY_i)\, e^{-qi} - D_i e^{-ri}. \tag{40}$$

  - Y_i is the yen inflow at year i.
  - S is the \$/yen spot exchange rate.
  - q is the yen interest rate.
  - D_i is the dollar outflow at year i.
  - r is the dollar interest rate.

As a Package of Forward Contracts (concluded)

- This formulation may be preferred to the cash market approach in cases involving costs of carry and convenience yields because forward prices already incorporate them.
- For simplicity, flat term structures were assumed.
- Generalization is straightforward.

Example

- Take the swap in the example on p. 386.
- Every year, B receives 11 million yen and pays 0.115 million dollars.
- In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars.
- Each of these transactions represents a forward contract.
- $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$.
- Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars as before.

Stochastic Processes and Brownian Motion

Of all the intellectual hurdles which the human mind has confronted and has overcome in the last fifteen hundred years, the one which seems to me to have been the most amazing in character and the most stupendous in the scope of its consequences is the one relating to the problem of motion.
— Herbert Butterfield (1900–1979)

Stochastic Processes

- A stochastic process

$$X = \{ X(t) \}$$

is a time series of random variables.

- $X(t)$ (or X_t) is a random variable for each time t and is usually called the state of the process at time t.
- A realization of X is called a sample path.
- A sample path defines an ordinary function of t.

Stochastic Processes (concluded)

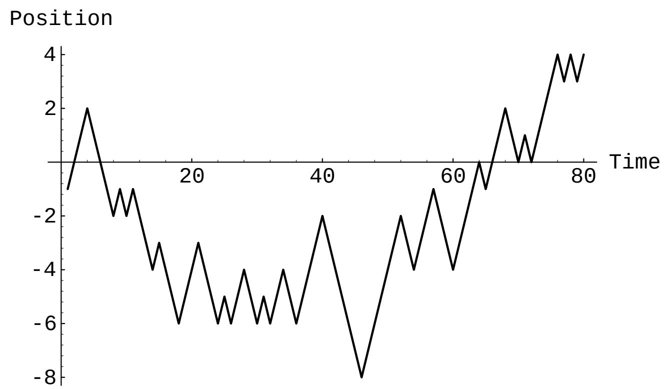
- If the times t form a countable set, X is called a discrete-time stochastic process or a time series.
- In this case, subscripts rather than parentheses are usually employed, as in

$$X = \{ X_n \}.$$

- If the times form a continuum, X is called a continuous-time stochastic process.

Random Walks

- The binomial model is a random walk in disguise.
- Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$.
- In each time step, it can make one move to the right with probability p or one move to the left with probability $1 - p$.
  - This random walk is symmetric when $p = 1/2$.
- Connection with the BOPM: The particle's position denotes the cumulative number of up moves minus that of down moves.



Random Walk with Drift

$$X_n = \mu + X_{n-1} + \xi_n.$$

- ξ_n are independent and identically distributed with zero mean.
- Drift μ is the expected change per period.
- Note that this process is continuous in space.

Martingales[a]

- $\{ X(t), t \geq 0 \}$ is a martingale if $E[\,|\,X(t)\,|\,] < \infty$ for $t \geq 0$ and
$$E[\,X(t)\,|\,X(u), 0 \leq u \leq s\,] = X(s). \tag{41}$$
- In the discrete-time setting, a martingale means
$$E[\,X_{n+1}\,|\,X_1, X_2, \ldots, X_n\,] = X_n. \tag{42}$$
- X_n can be interpreted as a gambler's fortune after the nth gamble.
- Identity (42) then says the expected fortune after the $(n+1)$th gamble equals the fortune after the nth gamble regardless of what may have occurred before.

[a]The origin of the name is somewhat obscure.

Martingales (concluded)

- A martingale is therefore a notion of fair games.
- Apply the law of iterated conditional expectations to both sides of Eq. (42) on p. 397 to yield
$$E[\,X_n\,] = E[\,X_1\,] \tag{43}$$
for all n.
- Similarly, $E[\,X(t)\,] = E[\,X(0)\,]$ in the continuous-time case.

Still a Martingale?

- Suppose we replace Eq. (42) on p. 397 with
$$E[\,X_{n+1}\,|\,X_n\,] = X_n.$$
- It also says past history cannot affect the future.
- But is it equivalent to the original definition?[a]

[a]Contributed by Mr. Hsieh, Chicheng (xxx) on April 13, 2005.

Still a Martingale? (continued)

- Well, no.[a]
- Consider this random walk with drift:

$$X_i = \begin{cases} X_{i-1} + \xi_i, & \text{if } i \text{ is even,} \\ X_{i-2}, & \text{otherwise.} \end{cases}$$

- Above, ξ_n are random variables with zero mean.

[a] Contributed by Mr. Zhang, Ann-Sheng (B89201033) on April 13, 2005.

Still a Martingale? (concluded)

- It is not hard to see that

$$E[\, X_i \,|\, X_{i-1} \,] = \begin{cases} X_{i-1}, & \text{if } i \text{ is even,} \\ X_{i-1}, & \text{otherwise.} \end{cases}$$

- Hence it is a martingale by the "new" definition.
- But

$$E[\, X_i \,|\, \ldots, X_{i-2}, X_{i-1} \,] = \begin{cases} X_{i-1} & \text{if } i \text{ is even,} \\ X_{i-2} & \text{otherwise.} \end{cases}$$

- Hence it is not a martingale by the original definition.