Sensitivity Analysis of Options

Cleopatra’s nose, had it been shorter, the whole face of the world would have been changed. — Blaise Pascal (1623–1662)

Sensitivity Measures ("The Greeks")

- Understanding how the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- Let \( x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}} \) (recall p. 239).
- Note that \( N'(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} > 0 \), the density function of standard normal distribution.

Delta

- Defined as \( \Delta \equiv \partial f / \partial S \).
  - \( f \) is the price of the derivative.
  - \( S \) is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.
- The delta used in the BOPM is the discrete analog.
Delta (concluded)

- The delta of a European call on a non-dividend-paying stock equals
  \[ \frac{\partial C}{\partial S} = N(x) > 0. \]
- The delta of a European put equals
  \[ \frac{\partial P}{\partial S} = N(x) - 1 < 0. \]
- The delta of a long stock is 1.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
- A delta-neutral portfolio is immune to small price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and \(-\Delta\) shares of stock is delta-neutral.
  - Short \(\Delta\) shares of stock to hedge a long call.
- In general, hedge a position in a security with a delta of \(\Delta_1\) by shorting \(\Delta_1/\Delta_2\) units of a security with a delta of \(\Delta_2\).

Theta (Time Decay)

- Defined as the rate of change of a security’s value with respect to time, or \(\Theta \equiv -\frac{\partial f}{\partial \tau}\).
- For a European call on a non-dividend-paying stock,
  \[ \Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} - rX e^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0. \]
  - The call loses value with the passage of time.
- For a European put,
  \[ \Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} + rX e^{-r\tau}N(-x + \sigma\sqrt{\tau}). \]
  - Can be negative or positive.

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or \(\Gamma \equiv \frac{\partial^2 \Pi}{\partial S^2}\).
- Measures how sensitive delta is to changes in the price of the underlying asset.
- A portfolio with a high gamma needs in practice be rebalanced more often to maintain delta neutrality.
- Delta \(\sim\) duration; gamma \(\sim\) convexity.
- The gamma of a European call or put on a non-dividend-paying stock is
  \[ \frac{N'(x)}{(S\sigma\sqrt{\tau})} > 0. \]
Vega (Lambda, Kappa, Sigma)

- Defined as the rate of change of its value with respect to the volatility of the underlying asset, \( \Lambda \equiv \frac{\partial \Pi}{\partial \sigma} \).
- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes in volatility.
- The vega of a European call or put on a non-dividend-paying stock is \( S \sqrt{\tau} N'(x) > 0 \).
  - Higher volatility increases option value.

\[ a \text{ Vega is not Greek.} \]

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,
  \[ \frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S} \]
- The computation time roughly doubles that for evaluating the derivative security itself.

Rho

- Defined as the rate of change in its value with respect to interest rates, \( \rho \equiv \frac{\partial \Pi}{\partial r} \).
- The rho of a European call on a non-dividend-paying stock is \( X \tau e^{-r \tau} N(x - \sigma \sqrt{\tau}) > 0 \).
- The rho of a European put on a non-dividend-paying stock is \( -X \tau e^{-r \tau} N(-x + \sigma \sqrt{\tau}) < 0 \).

\[ a \text{ A standard method computes the finite difference,} \]

An Alternative Numerical Delta

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, \( f_u \) and \( f_d \) are computed.
- These values correspond to derivative values at stock prices \( S_u \) and \( S_d \), respectively.
- Delta is approximated by \( \frac{f_u - f_d}{S_u - S_d} \).
- Almost zero extra computational effort.

\[ a \text{ Pelsser and Vorst (1994).} \]
Numerical Gamma

- At the stock price \((Suu + Sud)/2\), delta is approximately \((f_{uu} - f_{ud})/(Suu - Sud)\).
- At the stock price \((Sud + Sdd)/2\), delta is approximately \((f_{ud} - f_{dd})/(Sud - Sdd)\).
- Gamma is the rate of change in deltas between \((Suu + Sud)/2\) and \((Sud + Sdd)/2\), that is,

\[
\frac{f_{uu} - f_{ud}}{Suu - Sud} - \frac{f_{ud} - f_{dd}}{Sud - Sdd}
\]

\(\frac{(Suu - Sdd)^2}{(Suu - Sdd)/2}\).

Other Numerical Greeks

- The theta can be computed as

\[
\frac{f_{ud} - f}{2(\tau/n)}
\]

- In fact, the theta of a European option will be shown to be computable from delta and gamma (see p. 502).
- For vega and rho, there is no alternative but to run the binomial tree algorithm twice.

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

\[
\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}
\]

- It does not work (see text).
- Why did the binomial tree version work?
Extensions of Options Theory

As I never learnt mathematics, so I have had to think.
— Joan Robinson (1903–1983)

Pricing Corporate Securities

• Interpret the underlying asset interpreted as the total value of the firm.
• The option pricing methodology can be applied to pricing corporate securities.
• Assume:
  – A firm can finance payouts by the sale of assets.
  – If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

*Risk and Corporate Securities (1973).*

Risky Zero-Coupon Bonds and Stock

• Consider XYZ.com.
• Capital structure:
  – $n$ shares of its own common stock, $S$.
  – Zero-coupon bonds with an aggregate par value of $X$.
• What is the value of the bonds, $B$?
• What is the value of the XYZ.com stock?
Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds’ maturity date, suppose the total value of the firm $V^*$ is less than the bondholders’ claim $X$.
- Then the firm declares bankruptcy and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain $X$ and the stockholders $V^* - X$.

\[
\begin{array}{ccc}
V^* & \leq X & V^* > X \\
\text{Bonds} & V^* & X \\
\text{Stock} & 0 & V^* - X
\end{array}
\]

Risky Zero-Coupon Bonds and Stock (continued)

- Thus $nS = C$ and $B = V - C$.
- Knowing $C$ amounts to knowing how the value of the firm is divided between the stockholders and the bondholders.
- Whatever the value of $C$, the total value of the stock and bonds at maturity remains $V^*$.
- The relative size of debt and equity is irrelevant to the firm’s current value $V$.

\[
\begin{align*}
nS &= V N(x) - X e^{-r\tau} N(x - \sigma\sqrt{\tau}) , \\
B &= V N(-x) + X e^{-r\tau} N(x - \sigma\sqrt{\tau}) .
\end{align*}
\]

- where
\[
x = \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} .
\]
- The continuously compounded yield to maturity of the firm’s bond is
\[
\frac{\ln(X/B)}{\tau} .
\]
Risky Zero-Coupon Bonds and Stock (concluded)

- Define default premium as the yield difference between risky and riskless bonds,

\[
\frac{1}{\tau} \ln \left( \frac{X}{B} \right) - r = - \frac{1}{\tau} \ln \left( N(-z) + \frac{1}{\omega} N(z - \sigma \sqrt{\tau}) \right).
\]

- \( \omega \equiv X e^{-r \tau} / V. \)
- \( z \equiv (\ln \omega)/(\sigma \sqrt{\tau}) + (1/2) \sigma \sqrt{\tau} = -x + \sigma \sqrt{\tau}. \)
- Note that \( \omega \) is the debt-to-total-value ratio.

A Numerical Example

- XYZ.com’s assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck’s market value per share is $44.5.
- XYZ.com’s securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay $1,000 at maturity.
- \( n = 1000, \ V = 44.5 \times n = 44500, \) and \( X = 30 \times n = 30000. \)

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of \( X/n = 30 \) dollars.
- Such a call is selling for $15.25.
- So XYZ.com’s stock is worth \( 15.25 \times n = 15250 \) dollars.
- The entire bond issue is worth \( B = 44500 - 15250 = 29250 \) dollars.
  - Or $975 per bond.
A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $X$ par value plus $n$ written European puts on Merck at a strike price of $30$.
  - By the put-call parity.
- The difference between $B$ and the price of the default-free bond is the value of these puts.
- The table next page shows the total market values of the XYZ.com stock and bonds under various debt amounts $X$.

---

<table>
<thead>
<tr>
<th>Promised payment to bondholders</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>30,000</td>
</tr>
<tr>
<td>35,000</td>
</tr>
<tr>
<td>40,000</td>
</tr>
<tr>
<td>45,000</td>
</tr>
</tbody>
</table>

A Numerical Example (continued)

- Suppose the promised payment to bondholders is $45,000$.
- Then the relevant option is the July call with a strike price of $45000/n = 45$ dollars.
- Since that option is selling for $115/16$, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option’s terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
- So parameters such volatility, dividend, and strike price are under partial control of the stockholders.
A Numerical Example (continued)

• Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
• The total debt is now $X = 45,000$ dollars.
• The table on p. 286 says the total market value of the bonds should be $42,562.5$.
• The new bondholders pay $42562.5 \times (15/45) = 14187.5$ dollars.
• The remaining stock is worth $1,937.5$.

A Numerical Example (continued)

• The stockholders therefore gain $14187.5 + 1937.5 - 15250 = 875$ dollars.
• The original bondholders lose an equal amount,
  \[ 29250 - \frac{30}{45} \times 42562.5 = 875. \] (28)

A Numerical Example (continued)

• Suppose the stockholders distribute $14,833.3$ cash dividends by selling $(1/3) \times n$ Merck shares.
• They now have $14,833.3$ in cash plus a call on $(2/3) \times n$ Merck shares.
• The strike price remains $X = 30000$.
• This is equivalent to owning two-thirds of a call on $n$ Merck shares with a total strike price of $45,000$.
• The $n$ such calls are worth $1,937.5$ (p. 283).
• So the total market value of the XYZ.com stock is $(2/3) \times 1937.5 = 1291.67$ dollars.

A Numerical Example (concluded)

• The market value of the XYZ.com bonds is hence $(2/3) \times n \times 44.5 - 1291.67 = 28375$ dollars.
• Hence the stockholders gain $14833.3 + 1291.67 - 15250 \approx 875$ dollars.
• The bondholders watch their value drop from $29,250$ to $28,375$, a loss of $875$. 

©2005 Prof. Yuh-Dauh Lyuu, National Taiwan University
Other Examples

- Subordinated debts as bull call spreads.
- Warrants as calls.
- Callable bonds as American calls with 2 strike prices.
- Convertible bonds.

Barrier Options

- A knock-in option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all kinds of barrier options.

A former student told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank.

Barrier Options (concluded)

- A knock-out option is an ordinary European option which ceases to exist if the barrier $H$ is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.
S = 8, \( X = 6 \), \( H = 4 \), \( R = 1.25 \), \( u = 2 \), and \( d = 0.5 \).
Backward-induction: \( C = (0.5 \times C_u + 0.5 \times C_d)/1.25 \).

Binomial Tree Algorithms (concluded)

- But convergence is erratic because \( H \) is not at a price level on the tree (see plot on next page).
  - Typically, the barrier has to be adjusted to be at a price level.
- Solutions will be presented later.

Daily Monitoring

- Almost all barrier options monitor the barrier only for the daily closing prices.
- In that case, only nodes at the end of a day need to check for the barrier condition.
- We can even remove intraday nodes to create a multinomial tree.
  - A node is then followed by \( d + 1 \) nodes if each day is partitioned into \( d \) periods.
- This saves time and space.
Foreign Currencies
- $S$ denotes the spot exchange rate in domestic/foreign terms.
- $\sigma$ denotes the volatility of the exchange rate.
- $r$ denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.
- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a “continuous dividend yield” equal to $\hat{r}$ in the foreign currency.

Foreign Exchange Options
- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)
- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases $100,000,000/6,250,000 = 16$ puts on the Japanese yen with a strike price of $.0088 and an exercise month in March 2000.
- This gives the company the right to sell 100,000,000 Japanese yen for $100,000,000 \times .0088 = 880,000$ U.S. dollars.
Foreign Exchange Options (concluded)

- The formulas derived for stock index options in Eqs. (26) on p. 255 apply with the dividend yield equal to $\hat{r}$:

$$
C = S e^{-\hat{r} \tau} N(x) - X e^{-r \tau} N(x - \sigma \sqrt{\tau}),
$$

$$
P = X e^{-r \tau} N(-x + \sigma \sqrt{\tau}) - S e^{-\hat{r} \tau} N(-x),
$$

(29)

(29')

- where

$$
x = \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}.
$$

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.

- $S_0$ is the known price at time zero.

- $S_n$ is the price at expiration.

- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.

- Its value thus depends only on the underlying asset’s terminal price regardless of how it gets there.

Path-Dependent Derivatives (continued)

- In contrast, some derivatives are path-dependent in that their terminal payoff depends “critically” on the path.

- The (arithmetic) average-rate call has a terminal value given by

$$
\max \left( \frac{1}{n+1} \sum_{i=0}^{n} S_i - X, 0 \right).
$$

- The average-rate put’s terminal value is given by

$$
\max \left( X - \frac{1}{n+1} \sum_{i=0}^{n} S_i, 0 \right).
$$
Path-Dependent Derivatives (continued)

- Average-rate options are also called Asian options.
- They are very popular.\(^a\)
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.
- Like painting on a canvas, each brush stroke leaves less room for the future composition.

\(^a\)As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars; see Nielsen and Sandmann (2003).

Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine.
- A straightforward algorithm is to enumerate the \(2^n\) price paths for an \(n\)-period binomial tree and then average the payoffs.
- But the exponential complexity makes this naive algorithm impractical.
- As a result, the Monte Carlo method and approximation algorithms are some of the alternatives left.

Path-Dependent Derivatives (concluded)

- A lookback call option on the minimum has a terminal payoff of \(S_n - \min_{0 \leq i \leq n} S_i\).
- A lookback put option on the maximum has a terminal payoff of \(\max_{0 \leq i \leq n} S_i - S_n\).
- The fixed-strike lookback option provides a payoff of \(\max(\max_{0 \leq i \leq n} S_i - X, 0)\) for the call and \(\max(X - \min_{0 \leq i \leq n} S_i, 0)\) for the put.
- Lookback call and put options on the average are called average-strike options.
Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
- Barrier options are easy to price.
- When averaging is done geometrically, the option payoffs are

\[
\begin{align*}
\max \left( (S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0 \right), \\
\max \left( X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0 \right) 
\end{align*}
\]

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas.
  - With the volatility set to \( \sigma_a \equiv \sigma / \sqrt{3} \).
  - With the dividend yield set to \( q_a \equiv (r + q + \sigma^2 / 6) / 2 \).
- The formula is therefore

\[
\begin{align*}
C &= Se^{-q_a \tau} N(x) - X e^{-r \tau} N(x - \sigma_a \sqrt{\tau}), \\
P &= X e^{-r \tau} N(-x) - S e^{-q_a \tau} N(-x) 
\end{align*}
\]

\[
\begin{align*}
C &= Se^{-q_a \tau} N(x) - X e^{-r \tau} N(x - \sigma_a \sqrt{\tau}), \\
P &= X e^{-r \tau} N(-x) - S e^{-q_a \tau} N(-x) 
\end{align*}
\]

- where \( x \equiv \frac{\ln(S/X) + (r - q_a + \sigma_a^2 / 2) \tau}{\sigma_a \sqrt{\tau}} \).