The Setting

- The no-arbitrage principle is insufficient to pin down the exact option value without further assumptions on the probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option’s payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.
- Known as the Black-Scholes option pricing model.

Terms and Approach

- $C$: call value.
- $P$: put value.
- $X$: strike price
- $S$: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.
Binomial Option Pricing Model (BOPM)

- Time is discrete and measured in periods.
- If the current stock price is $S$, it can go to $Su$ with probability $q$ and $Sd$ with probability $1-q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d < R < u$ must hold to rule out arbitrage.
- Six pieces of information suffice to determine the option value based on arbitrage considerations: $S$, $u$, $d$, $X$, $\hat{r}$, and the number of periods to expiration.

Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- $C_u$ is the call price at time one if the stock price moves to $Su$.
- $C_d$ is the call price at time one if the stock price moves to $Sd$.
- Clearly,
  
  $$
  C_u = \max(0, Su - X),
  
  C_d = \max(0, Sd - X).
  $$

\[\begin{array}{c}
S \\
qu \quad Su \\
1-q \quad Sd
\end{array}\]

\[\begin{array}{c}
C \\
q \quad \quad C_u = \max(0, Su - X) \\
1-q \quad C_d = \max(0, Sd - X)
\end{array}\]
Call on a Non-Dividend-Paying Stock: Single Period
(continued)

- Set up a portfolio of $h$ shares of stock and $B$ dollars in riskless bonds.
  - This costs $hS + B$.
  - We call $h$ the hedge ratio or delta.
- The value of this portfolio at time one is either $hSu + RB$ or $hSd + RB$.
- Choose $h$ and $B$ such that the portfolio replicates the payoff of the call,
  \[
  hSu + RB = C_u, \\
  hSd + RB = C_d.
  \]

Call on a Non-Dividend-Paying Stock: Single Period
(concluded)

- Solve the above equations to obtain
  \[
  h = \frac{C_u - C_d}{Su - Sd} \geq 0, \quad (20) \\
  B = \frac{uC_d - dC_u}{(u-d)R} \quad (21)
  \]
- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio, $C = hS + B$.
- As $uC_d - dC_u < 0$, the equivalent portfolio is a levered long position in stocks.

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 3 (p. 182), so $C = hS + B$.

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where
  \[
  P_u = \max(0, X - Su), \\
  P_d = \max(0, X - Sd).
  \]
- Let $B = \frac{uP_d - dP_u}{(u-d)R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$. 
Risk

- Surprisingly, the option value is independent of $q$.
- Hence it is independent of the expected gross return of the stock, $qS_u + (1 - q)S_d$.
- It therefore does not directly depend on investors’ risk preferences.
- The option value does depend on the sizes of price changes, $u$ and $d$, the magnitudes of which the investors must agree upon.

Pseudo Probability

- After substitution and rearrangement,
  \[ hS + B = \frac{(R-d)}{u-d} C_u + \frac{(u-R)}{u-d} C_d \]
  \[ \frac{R}{n} \]  \( (22) \)
- Rewrite Eq. (22) as
  \[ hS + B = \frac{pC_u + (1 - p) C_d}{R} \]
  where
  \[ p = \frac{R - d}{u - d} \]
  - As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under $q = p$ as $pS_u + (1 - p)S_d = RS$.
- Risk-neutral investors care only about expected returns.
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, $p$ is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the FV is the riskless rate in a risk-neutral economy.

Binomial Distribution

- Denote the binomial distribution with parameters $n$ and $p$ by
  \[ b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \]
  - $n! = n \times (n-1) \cdots 2 \times 1$ with the convention $0! = 1$.
- Suppose you toss a coin $n$ times with $p$ being the probability of getting heads.
- Then $b(j; n, p)$ is the probability of getting $j$ heads.
Option on a Non-Dividend-Paying Stock: Multi-Period

• Consider a call with two periods remaining before expiration.
• Under the binomial model, the stock can take on three possible prices at time two: $S_{uu}$, $S_{ud}$, and $S_{dd}$.
  - Note that the tree combines.
• At any node, the next two stock prices only depend on the current price, not the prices of earlier times.
• This memoryless property is a key feature of an efficient market.

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

• Let $C_{uu}$ be the call’s value at time two if the stock price is $S_{uu}$.
• Thus,
  $$C_{uu} = \max(0, S_{uu} - X).$$
• $C_{ud}$ and $C_{dd}$ can be calculated analogously,
  $$C_{ud} = \max(0, S_{ud} - X),$$
  $$C_{dd} = \max(0, S_{dd} - X).$$
Option on a Non-Dividend-Paying Stock: Multi-Period
(continued)

- The call values at time one can be obtained by applying
  the same logic:

  \[ C_u = \frac{pC_{uu} + (1 - p)C_{ud}}{R}, \]
  \[ C_d = \frac{pC_{ud} + (1 - p)C_{dd}}{R}. \] (23)
- Deltas can be derived from Eq. (20) on p. 196.
- For example, the delta at \( C_u \) is
  \[ \left( C_{uu} - C_{ud} \right) / \left( Suu - Sud \right). \]

Early Exercise

- Since the call will not be exercised at time one even if it is
  American, \( C_u \ge Su - X \) and \( C_d \ge Sd - X \).
- Therefore,
  \[ hS + B = \frac{pC_u + (1 - p)C_d}{R} \ge \left[ \frac{pu + (1 - p)d}{R} \right] S - X \]
  \[ = S - \frac{X}{R} > S - X. \]
- So the call again will not be exercised at present, and
  \[ C = hS + B = \frac{pC_u + (1 - p)C_d}{R}. \]

Option on a Non-Dividend-Paying Stock: Multi-Period
(concluded)

- We now reach the current period.
- An equivalent portfolio of \( h \) shares of stock and \$B
  riskless bonds can be set up for the call that costs \( C_u \)
  \( (C_d, \text{resp.}) \) if the stock price goes to \( Su \) \( (Sd, \text{resp.}) \).
- The values of \( h \) and \( B \) can be derived from
  Eqs. (20)–(21) on p. 196.
- Or, we can just compute
  \[ \frac{pC_u + (1 - p)C_d}{R} \]
  as the price.

Backward Induction of Zermelo (1871–1953)

- The above expression calculates \( C \) from the two
  successor nodes \( C_u \) and \( C_d \) and none beyond.
- The same computation happens at \( C_u \) and \( C_d \), too, as
  demonstrated in Eq. (23) on p. 207.
- This recursive procedure is called backward induction.
- Now, \( C \) equals
  \[ \left[ p^2C_{uu} + 2p(1 - p)C_{ud} + (1 - p)^2C_{dd} \right] / R^2 \]
  \[ = \left[ p^2 \max \left( 0, Su^2 - X \right) + 2p(1 - p) \max \left( 0, Sud - X \right) \right. \]
  \[ \left. + (1 - p)^2 \max \left( 0, Sd^2 - X \right) \right] / R^2. \]
**Backward Induction (continued)**

- In the $n$-period case,
  
  \[
  C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max (0, Su^j d^{n-j} - X)}{R^n}.
  \]

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- The value of a European put is
  \[
  P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max (0, X - Su^j d^{n-j})}{R^n}.
  \]

**Risk-Neutral Pricing Methodology**

- Every derivative can be priced as if the economy were risk-neutral.

- For a European-style derivative with the terminal payoff function $D$, its value is
  \[
  e^{-\tilde{r}n} E^{\pi} [D].
  \]

  - $E^{\pi}$ means the expectation is taken under the risk-neutral probability.

- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

**Self-Financing**

- Delta changes over time.

- The maintenance of an equivalent portfolio is dynamic.

- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.

- The portfolio’s value at the end of the current period is precisely the amount needed to set up the next portfolio.

- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.

  - Changes in value are due entirely to capital gains.
The Binomial Option Pricing Formula

- Let $a$ be the minimum number of upward price moves for the call to finish in the money.
- So $a$ is the smallest nonnegative integer such that
  \[ Su^a d^{n-a} \geq X, \]
  or
  \[ a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil. \]

The Binomial Option Pricing Formula (concluded)

Hence,
\[
C = \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} (Su^j d^{n-j} - X)}{R^n} \tag{24}
\]
\[
= S \sum_{j=a}^{n} \left( \binom{n}{j} (pu)^j (1-p)^{n-j} \right) - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j}
\]
\[
= S \sum_{j=a}^{n} b(j; n, pue^{-r}) - X e^{-rn} \sum_{j=a}^{n} b(j; n, p).
\]

Numerical Examples

- A non-dividend-paying stock is selling for $160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is $85.069$ by backward induction.
- Also the PV of the expected payoff at expiration,
  \[
  \frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.
  \]
Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for $90 instead.
- Sell the call for $90 and invest $85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,
  
  \[ 90 - 85.069 = 4.931 \text{ dollars}, \]

  is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 1:
- Suppose the stock price moves to $240.
- The new delta is 0.90625.
- Buy $0.90625 - 0.82031 = 0.08594$ more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

Time 2:
- Suppose the stock price plunges to $120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to $76.04232 \times 1.2 - 78.75 = 12.5$ dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):
- The stock price moves to $180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.
Numerical Examples (concluded)

Time 3 (the case of declining price):
- The stock price moves to $60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of $0.25 \times 60 = 15$ dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n^2)$.
  - Can be further reduced to $O(n)$ by reusing space
- To price European puts, simply replace the payoff.
Optimal Algorithm

• We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

• Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p) j} b(j - 1; n, p).$$

• The following program computes $b(j; n, p)$ in $b[j],$

1: $b[a] := \binom{n}{a} p^a (1 - p)^{n-a};$
2: for $j = a + 1, a + 2, \ldots, n$ do
3: $b[j] := b[j - 1] \times p \times (n - j + 1) / ((1 - p) \times j);$
4: end for

• It runs in $O(n)$ steps.

Optimal Algorithm (concluded)

• With the $b(j; n, p)$ available, the risk-neutral valuation formula (24) on p. 216 is trivial to compute.

• We only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.

• This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.

• The above technique cannot be applied to American options because of early exercise.

• So binomial tree algorithms for American options usually run in $O(n^2)$ time.

On the Bushy Tree

Toward the Black-Scholes Formula

• The binomial model suffers from two unrealistic assumptions.
  – The stock price takes on only two values in a period.
  – Trading occurs at discrete points in time.

• As the number of periods increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.

• Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.

• We now skim through the proof.
Toward the Black-Scholes Formula (continued)

• Let $\tau$ denote the time to expiration of the option measured in years.
• Let $r$ be the continuously compounded annual rate.
• With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.
• Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n$ goes to infinity.
  - First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

• Assume the stock’s true continuously compounded rate of return over $\tau$ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
  - Call $\sigma$ the stock’s (annualized) volatility.
• The BOPM converges to the distribution only if
  \[ n\hat{\mu} = n(q\ln(u/d) + \ln d) \rightarrow \mu\tau, \]
  \[ n\hat{\sigma}^2 = nq(1 - q)\ln^2(u/d) \rightarrow \sigma^2\tau. \]
• Impose $ud = 1$ to make nodes at the same horizontal level of the tree have identical price (review p. 226).
  - Other choices are possible (see text).

Toward the Black-Scholes Formula (continued)

• Use $\hat{\mu} \equiv \frac{1}{n} E \left[ \ln \frac{S_\tau}{S} \right]$ and $\hat{\sigma}^2 \equiv \frac{1}{n} \text{Var} \left[ \ln \frac{S_\tau}{S} \right]$ to denote, resp., the expected value and variance of the period continuously compounded rate of return.
• Under the BOPM, it is not hard to show that
  \[ \hat{\mu} = q\ln(u/d) + \ln d, \]
  \[ \hat{\sigma}^2 = q(1 - q)\ln^2(u/d). \]

Toward the Black-Scholes Formula (continued)

• The above requirements can be satisfied by
  \[ u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\tau/n}. \quad (25) \]
  - With Eqs. (25),
    \[ n\hat{\mu} = \mu\tau, \]
    \[ n\hat{\sigma}^2 = \left[ 1 - \left( \frac{\mu}{\sigma} \right)^2 \frac{\tau}{n} \right] \sigma^2\tau \rightarrow \sigma^2\tau. \]
  - Other choices are possible (see text).
Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $u > R > d$ may not hold under Eqs. (25).
- If this happens, the risk-neutral probability may lie outside $[0, 1]$.
- The problem disappears when $n$ satisfies
  \[ e^{\sigma \sqrt{\tau/n}} > e^{r\tau/n}, \]
  in other words, when $n > r^2\tau/\sigma^2$.
  - So it goes away if $n$ is large enough.
  - Other solutions will be presented later.

---

Toward the Black-Scholes Formula (continued)

**Lemma 7** The continuously compounded rate of return \( \ln(S_T/S) \) approaches the normal distribution with mean \( (r - \sigma^2/2)\tau \) and variance \( \sigma^2\tau \) in a risk-neutral economy.

- Let \( q \) equal the risk-neutral probability
  \[ p \equiv (e^{r\tau/n} - d)/(u - d). \]
- Let \( n \to \infty \).

---

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return \( \ln(S_T/S) \)?
- The central limit theorem says \( \ln(S_T/S) \) converge to the normal distribution with mean \( \mu\tau \) and variance \( \sigma^2\tau \).
- So \( \ln S_T \) approaches the normal distribution with mean \( \mu\tau + \ln S \) and variance \( \sigma^2\tau \).
- \( S_T \) has a lognormal distribution in the limit.

---

Toward the Black-Scholes Formula (continued)

- Lemma 7 and Eq. (18) on p. 144 imply the expected stock price at expiration in a risk-neutral economy is \( Se^{r\tau} \).
- The stock’s expected annual rate of return\(^a\) is thus the riskless rate \( r \).

\(^a\)In the sense of \((1/\tau)\ln E[S_T/S] \) not \((1/\tau)E[\ln(S_T/S)]\).
Toward the Black-Scholes Formula (concluded)

Theorem 8 (The Black-Scholes Formula)

\[ C = SN(x) - X e^{-r \tau} N(x - \sigma \sqrt{\tau}) , \]
\[ P = X e^{-r \tau} N(-x + \sigma \sqrt{\tau}) - SN(-x) , \]

where

\[ x \equiv \ln(S/X) + \left(r + \sigma^2/2\right) \tau / \sigma \sqrt{\tau}. \]

BOPM and Black-Scholes Model

- The Black-Scholes formula needs five parameters: \( S \), \( X \), \( \sigma \), \( \tau \), and \( r \).
- Binomial tree algorithms take six inputs: \( S \), \( X \), \( u \), \( d \), \( \hat{r} \), and \( n \).
- The connections are
  \[ u = e^{\sigma \sqrt{\tau/n}}, \quad d = e^{-\sigma \sqrt{\tau/n}}, \quad \hat{r} = r \tau / n. \]
- The binomial tree algorithms converge reasonably fast.
- Oscillations can be eliminated by the judicious choices of \( u \) and \( d \) (see text).

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market’s opinion of the volatility.
  - Solve for \( \sigma \) given the option price, \( S \), \( X \), \( \tau \), and \( r \) with numerical methods.
  - How about American options?
- This volatility is called the implied volatility.
- Implied volatility is often preferred to historical volatility in practice.\(^a\)

\(^a\)It is like driving a car with your eyes on the rearview mirror?
Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a “smile” in relation to the strike price.
  - The implied volatility is lowest for at-the-money options and becomes higher the further the option is in- or out-of-the-money.

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But \( \sigma \) is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.\(^a\)
- How to incorporate these two different ways of day count into the Black-Scholes formula and binomial tree algorithms?

\(^a\)Fama (1965); French (1980); French and Roll (1986).

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Trading Days and Calendar Days (concluded)

- Suppose a year has 260 trading days.
- A quick and dirty way is to replace \( \sigma \) with\(^a\)
  \[
  \sigma \sqrt[260]{\frac{365}{\text{number of trading days to expiration}}} \cdot \sqrt[260]{\text{number of calendar days to expiration}}
  \]
- How about binomial tree algorithms?

\(^a\)French (1984).
Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
  \[
  \max(0, X - Su^{j}d^{n-j})
  \]
  and applies backward induction.
- At each intermediate node, it checks for early exercise by comparing the payoff if exercised with the continuation value.

Known Dividends

- Constant dividends introduce complications.
- Use \( D \) to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are \( Su - D \) and \( Sd - D \).
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: \( (Su - D)u \), \( (Su - D)d \), \( (Sd - D)u \), \( (Sd - D)d \).
  - The binomial tree no longer combines (see p. 229).

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.
- In general, the corporate dividend policy is a complex issue.
An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends (Roll, 1977).
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - $\sigma$ equal to the volatility of the process followed by the risky component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

An Uncompromising Approach

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases.

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  - A stock that grows from $S$ to $S_t$ with a continuous dividend yield of $q$ would grow from $S$ to $S_t e^{qt}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-qt}$ that pays no dividends.

---

$^a$Dai and Lyuu (2004).
Continuous Dividend Yields (continued)

- The Black-Scholes formulas hold with $S$ replaced by $Se^{-q\tau}$ (Merton, 1973):

$$C = Se^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \quad (26)$$
$$P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau}N(-x), \quad (26')$$

where

$$x \equiv \ln\left(\frac{S}{X}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\tau.$$

- Formulas (26) and (26') remain valid as long as the dividend yield is predictable.

- Replace $q$ with the average annualized dividend yield.

Continuous Dividend Yields (concluded)

- To run binomial tree algorithms, pick the risk-neutral probability as

$$\frac{e^{(r-q)\Delta t} - d}{u - d}, \quad (27)$$

where $\Delta t \equiv \tau/n$.

- Because the stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- The $u$ and $d$ remain unchanged.

- Other than the change in Eq. (27), binomial tree algorithms stay the same.