Term Structure Dynamics

- An n-period zero-coupon bond’s price can be computed by assigning $1 to every node at period n and then applying backward induction.
- Repeating this step for \( n = 1, 2, \ldots \), one obtains the market discount function implied by the tree.
- The tree therefore determines a term structure.
- It also contains a term structure dynamics as taking any node in the tree as the current state induces a binomial interest rate tree and, again, a term structure.
- It defines how the term structure evolves over time.

An Approximate Calibration Scheme

- Start with the implied one-period forward rates and then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today’s one-period spot rate.
- In general, let \( f_j \) denote the forward rate in period \( j \).
- This forward rate can be derived from the market discount function via \( f_j = (d(j)/d(j + 1)) - 1 \) (see Exercise 5.6.3 in text).

Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
- Assume the short rate volatility is such that \( v \equiv r_h/r_0 = 1.5 \), independent of time.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate (%)</td>
<td>4</td>
<td>4.2</td>
<td>4.3</td>
</tr>
<tr>
<td>One-period forward rate (%)</td>
<td>4</td>
<td>4.4</td>
<td>4.5</td>
</tr>
<tr>
<td>Discount factor</td>
<td>0.96154</td>
<td>0.92101</td>
<td>0.88135</td>
</tr>
</tbody>
</table>

An Approximate Calibration Scheme (continued)

- Since the \( i \)th short rate, \( 1 \leq i \leq j \), occurs with probability \( 2 \binom{j}{i} \binom{j-1}{i-1} \), this means

\[
\sum_{i=1}^{j} 2 \binom{j}{i} \binom{j-1}{i-1} r_j^i v_j^{i-1} = f_j.
\]
- Thus

\[
r_j = \left( \frac{2}{1 + v_j} \right)^j f_j. \tag{84}
\]
- The binomial interest rate tree is trivial to set up.
An Approximate Calibration Scheme (concluded)

• The ensuing tree for the sample term structure appears in figure next page.
• For example, the price of the zero-coupon bond paying $1 at the end of the third period is
  \[
  \frac{1}{8} \times \frac{1}{1.04} \times \left( \frac{1}{1.0552} \times \left( \frac{1}{1.0739} + \frac{1}{1.0552} \times \left( \frac{1}{1.0564} + \frac{1}{1.0564} \right) \right) \right)
  \]
  or 0.88155, which exceeds discount factor 0.88135.
• The tree is thus not calibrated.
• Indeed, this bias is inherent (see text).

Issues in Calibration

• The model prices generated by the binomial interest rate tree should match the observed market prices.
• Perhaps the most crucial aspect of model building.
• Treat the backward induction for the model price of the \( m \)-period zero-coupon bond as computing some function of the unknown baseline rate \( r_m \) called \( f(r_m) \).
• A root-finding method is applied to solve \( f(r_m) = P \) for \( r_m \) given the zero's price \( P \) and \( r_1, r_2, \ldots, r_{m-1} \).
• This procedure is carried out for \( m = 1, 2, \ldots, n \).
• Runs in cubic time, hopelessly slow.

Binomial Interest Rate Tree Calibration

• Calibration can be accomplished in quadratic time by the use of forward induction (Javashidjan, 1991).
• The scheme records how much $1 at a node contributes to the model price.
• This number is called the state price.
  – It stands for the price of a state contingent claim that pays $1 at that particular node (state) and 0 elsewhere.
• The column of state prices will be established by moving forward from time 1 to time  \( n \).
Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time $j$ and there are $j+1$ nodes.
  - The baseline rate for period $j$ is $r = r_j$.
  - The multiplicative ratio be $v = v_j$.
  - $P_1, P_2, \ldots, P_j$ are the state prices a period prior, corresponding to rates $r, rv, \ldots, rv^j$.
- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j-1)$-period zero-coupon bond.

Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for $r$ is guaranteed.
- The state prices at time $j$ can now be calculated (see figure (a) next page).
- We call a tree with these state prices a binomial state price tree (see figure (b) next page).
- The calibrated tree is depicted in on p. 734.

Binomial Interest Rate Tree Calibration (continued)

- One dollar at time $j$ has a known market value of $1/[1 + S(j)]^j$, where $S(j)$ is the $j$-period spot rate.
- Alternatively, this dollar has a present value of
  \[
  g(r) = \frac{P_1}{(1+r)} + \frac{P_2}{(1+rv)} + \frac{P_3}{(1+rv^2)} + \cdots + \frac{P_j}{(1+rv^j)}.
  \]
- So we solve
  \[
  g(r) = \frac{1}{[1 + S(j)]^j} \quad (85)
  \]
  for $r$.  

\[\begin{array}{c}
A & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
\downarrow \downarrow \downarrow & 4.00% & 4.4% & 4.5% \\
\uparrow & \uparrow & \uparrow \\
A & B & C & D & E \\
\downarrow & \downarrow & \downarrow \downarrow & \downarrow \\
\downarrow \downarrow \downarrow \downarrow \downarrow & 2.895% & 3.526% & 4.00% & 4.4% & 4.5% \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
A & B & C & D & E & F & G \\
\end{array}\]
A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.
- The baseline rate for the second period, $r_2$, satisfies
  \[
  \frac{0.480769}{1 + r_2} + \frac{0.480769}{1 + 1.5 \times r_2} = 0.92101.
  \]
  The result is $r_2 = 3.526\%$.
- This is used to derive the next column of state prices shown in figure (b) on p. 733 as 0.232197, 0.460505, and 0.228308.
- Their sum gives the correct market discount factor 0.92101.

Binomial Interest Rate Tree Calibration (concluded)

- The Newton-Raphson method can be used to solve for the $r$ in Eq. (85) on p. 731 as $g'(r)$ is easy to evaluate.
- The monotonicity and the convexity of $g(r)$ also facilitate root finding.
- The above idea is straightforward to implement.
- The total running time is $O(Cn^2)$, where $C$ is the maximum number of times the root-finding routine iterates, each consuming $O(n)$ work.
- With a good initial guess, the Newton-Raphson method converges in only a few steps (Lyu, 1999).
Spread of Nonbenchmark Bonds (continued)

- We illustrate the idea with an example.
- Start with the tree on p. 741,
- Consider a security with cash flow $C_i$ at time $i$ for $i = 1, 2, 3$,
- Its model price is $p(s)$, which is equal to

$$
\frac{1}{1.04 + s} \times \left[ C_1 + \frac{1}{2} \times \frac{1}{1.03526 + s} \times \left( C_2 + \frac{1}{2} \times \left( C_3 + \frac{1}{1.02855 + s} \times \left( C_4 + \frac{1}{1.01433 + s} \times \left( C_5 \right) \right) \right) \right) \right] +
$$

- Given a market price of $P$, the spread is the $s$ that solves $P = p(s)$.
Spread of Nonbenchmark Bonds (continued)

- The model price \( p(s) \) is a monotonically decreasing, convex function of \( s \).
- We will employ the Newton-Raphson root-finding method to solve \( p(s) - P = 0 \) for \( s \).
- But a quick look at the equation above reveals that evaluating \( p'(s) \) directly is infeasible.
- Fortunately, the tree can be used to evaluate both \( p(s) \) and \( p'(s) \) during backward induction.

\[ p_A'(s) = \frac{p_B'(s) + p_C'(s)}{2(1 + r + s)} - \frac{p_B(s) + p_C(s)}{2(1 + r + s)^2}. \]  

Spread of Nonbenchmark Bonds (continued)

- To compute \( p'_A(s) \) as well, node A calculates

- This is easy if \( p_B'(s) \) and \( p_C'(s) \) are also computed at nodes B and C.
- Apply the above procedure inductively to yield \( p(s) \) and \( p'(s) \) at the root (see p. 745).
- This is called the differential tree method.\(^a\)

\(^a\)Lyuu (1999).

Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate \( r \).
- In the process of computing the model price \( p(s) \), a price \( p_A(s) \) is computed at A.
- Prices computed at A’s two successor nodes B and C are discounted by \( r + s \) to obtain \( p_A(s) \) as follows,

\[ p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1 + r + s)}, \]

where \( c \) denotes the cash flow at A.
Spread of Nonbenchmark Bonds (continued)

- Let $C$ represent the number of times the tree is traversed, which takes $O(n^2)$ time.
- The total running time is $O(Cn^2)$.
- In practice $C$ is a small constant.
- The memory requirement is $O(n)$.

Spread of Nonbenchmark Bonds (concluded)

- Consider a three-year, 5% bond with a market price of 100.569.
- Assume the bond pays annual interest.
- The spread can be shown to be 50 basis points over the tree (see p. 749).
- Note that the idea of spread does not assume parallel shifts in the term structure.
- It also differs from the yield spread and static spread of the nonbenchmark bond over an otherwise identical benchmark bond.

<table>
<thead>
<tr>
<th>Number of partitions $m$</th>
<th>Running time (s)</th>
<th>Number of revisions</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>7.50</td>
<td>5</td>
<td>1400</td>
<td>35,630,610</td>
</tr>
<tr>
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<td>11.50</td>
<td>5</td>
<td>11500</td>
<td>4,185,570</td>
</tr>
<tr>
<td>2500</td>
<td>19.77</td>
<td>5</td>
<td>12500</td>
<td>49,129,860</td>
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<tr>
<td>3500</td>
<td>38.76</td>
<td>5</td>
<td>13500</td>
<td>577,444</td>
</tr>
<tr>
<td>4500</td>
<td>64.00</td>
<td>5</td>
<td>14500</td>
<td>65,662,360</td>
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<tr>
<td>5500</td>
<td>90.60</td>
<td>5</td>
<td>15500</td>
<td>75,687,700</td>
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<tr>
<td>6500</td>
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<td>5</td>
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<td>85,029,050</td>
</tr>
<tr>
<td>7500</td>
<td>176.11</td>
<td>5</td>
<td>17500</td>
<td>95,23,900</td>
</tr>
<tr>
<td>8500</td>
<td>230.75</td>
<td>5</td>
<td>18500</td>
<td>105,172,270</td>
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<tr>
<td>9500</td>
<td>285.71</td>
<td>5</td>
<td>19500</td>
<td>115,541</td>
</tr>
</tbody>
</table>

75MHz Sun SPARCstation 20.

Cash flows: 5 5 105
More Applications of the Differential Tree: Calibrating Black-Derman-Toy (in seconds)

<table>
<thead>
<tr>
<th>Number of years</th>
<th>Running time</th>
<th>Number of years</th>
<th>Running time</th>
<th>Running time</th>
</tr>
</thead>
<tbody>
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<td>600</td>
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<td>300</td>
<td>396,800</td>
<td>7500</td>
</tr>
<tr>
<td>900</td>
<td>16,398,800</td>
<td>500</td>
<td>296,800</td>
<td>7500</td>
</tr>
<tr>
<td>1200</td>
<td>23,198,800</td>
<td>600</td>
<td>2196,000</td>
<td>7500</td>
</tr>
<tr>
<td>1500</td>
<td>23,198,800</td>
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<td>23,198,800</td>
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<tr>
<td>1800</td>
<td>339,800</td>
<td>300</td>
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</tr>
<tr>
<td>2100</td>
<td>23,198,800</td>
<td>400</td>
<td>24,198,800</td>
<td>7500</td>
</tr>
<tr>
<td>2400</td>
<td>23,198,800</td>
<td>500</td>
<td>23,198,800</td>
<td>7500</td>
</tr>
<tr>
<td>2700</td>
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<td>396,800</td>
<td>7500</td>
</tr>
<tr>
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<td>23,198,800</td>
<td>400</td>
<td>24,198,800</td>
<td>7500</td>
</tr>
<tr>
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<td>500</td>
<td>23,198,800</td>
<td>7500</td>
</tr>
<tr>
<td>3600</td>
<td>23,198,800</td>
<td>600</td>
<td>23,198,800</td>
<td>7500</td>
</tr>
</tbody>
</table>

75MHz Sun SPARCstation 20, one period per year.

More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)

<table>
<thead>
<tr>
<th>American call</th>
<th>American put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of parts</td>
<td>Running time</td>
</tr>
<tr>
<td>100</td>
<td>0,004210</td>
</tr>
<tr>
<td>50</td>
<td>0,004210</td>
</tr>
<tr>
<td>20</td>
<td>0,004210</td>
</tr>
<tr>
<td>10</td>
<td>0,004210</td>
</tr>
<tr>
<td>12</td>
<td>0,004210</td>
</tr>
<tr>
<td>11</td>
<td>0,004210</td>
</tr>
<tr>
<td>10</td>
<td>0,004210</td>
</tr>
</tbody>
</table>

Intel 166MHz Pentium, running on Microsoft Windows 95.

Fixed-Income Options

- Consider a two-year 99 European call on the three-year, 5% Treasury.
- Assume the Treasury pays annual interest.
- From p. 753 the three-year Treasury’s price minus the $5 interest could be $102,046, $100,630, or $98,579 two years from now.
- Since these prices do not include the accrued interest, we should compare the strike price against them.
- The call is therefore in the money in the first two scenarios, with values of $3,046 and $1,630, and out of the money in the third scenario.
Fixed-Income Options (continued)
- The option value is calculated to be $1,458 on p. 753(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only if the Treasury is worth $98,579 without the accrued interest.
- The option value is computed to be $0.096 on p. 753(b).

Delta or Hedge Ratio
- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as
  \[ \frac{O_h - O_t}{P_h - P_t}. \]
- In the above \( P_h \) and \( P_t \) denote the bond prices if the short rate moves up and down, respectively.
- Similarly, \( O_h \) and \( O_t \) denote the option values if the short rate moves up and down, respectively.

Fixed-Income Options (concluded)
- The present value of the strike price is \( PV(X) = 99 \times 0.92101 = 91.18 \).
- The Treasury is worth \( B = 101,955 \).
- The present value of the interest payments during the life of the options is
  \[ PV(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275. \]
- The call and the put are worth \( C = 1,458 \) and \( P = 0.096 \), respectively.
- Hence the put-call parity is preserved:
  \[ C = P + B - PV(I) = PV(X). \]

Delta or Hedge Ratio (concluded)
- Since delta measures the sensitivity of the option value to changes in the underlying bond price, it shows how to hedge one with the other.
- Take the call and put on p. 753 as examples.
- Their deltas are:
  \[ \frac{0.774 - 2.258}{99,350 - 102,716} = 0.441, \]
  \[ \frac{0.200 - 0.000}{99,350 - 102,716} = -0.059, \]
- respectively.
Volatility Term Structures

- The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.
- Consider an \( n \)-period zero-coupon bond.
- First find its yield to maturity \( y_h \) (\( y_e \), respectively) at the end of the initial period if the rate rises (declines, respectively).
- The yield volatility for our model is defined as \( (1/2) \ln(y_h/y_e) \).

Volatility Term Structures (continued)

- For example, based on the tree on p. 734, the two-year zero’s yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.
- Its yield volatility is therefore
  \[
  \frac{1}{2} \ln \left( \frac{0.05289}{0.03526} \right) = 20.273\%.
  \]

Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.
- If the rate rises, the price of the zero one year from now will be
  \[
  \frac{1}{2} \times \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.
  \]
- Thus its yield is \( \sqrt{0.90096} - 1 = 0.053531 \).
- If the rate declines, the price of the zero one year from now will be
  \[
  \frac{1}{2} \times \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.
  \]

Volatility Term Structures (continued)

- Thus its yield is \( \sqrt{0.93225} - 1 = 0.0357 \).
- The yield volatility is hence
  \[
  \frac{1}{2} \ln \left( \frac{0.053531}{0.0357} \right) = 20.256\%,
  \]
  slightly less than the one-year yield volatility.
- This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.
- The procedure can be repeated for longer-term zeros to obtain their yield volatilities.
Volatility Term Structures (continued)

- We started with \( v_i \) and then derived the volatility term structure.
- In practice, the steps are reversed,
- The volatility term structure is supplied by the user along with the term structure,
- The \( v_i \) hence the short rate volatilities via Eq. (82) on p. 714 and the \( r_i \) are then simultaneously determined,
- The result is the Black-Derman-Toy model.

Volatility Term Structures (concluded)

- Suppose the user supplies the volatility term structure which results in \( (v_1, v_2, v_3, \ldots) \) for the tree.
- The volatility term structure one period from now will be determined by \( (v_2, v_3, \ldots) \) not \( (v_1, v_2, v_3, \ldots) \).
- The volatility term structure supplied by the user is hence not maintained through time.
- This issue will be addressed by other types of (complex) models.

Foundations of Term Structure Modeling
Terminology

- A period denotes a unit of elapsed time.
  - Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.
- Bonds will be assumed to have a par value of one unless stated otherwise.
- The time unit for continuous-time models will usually be measured by the year.

Standard Notations

The following notation will be used throughout.

- $t$: a point in time.
- $r(t)$: the one-period riskless rate prevailing at time $t$ for repayment one period later (the instantaneous spot rate, or short rate, at time $t$).
- $P(t, T)$: the present value at time $t$ of one dollar at time $T$.

Standard Notations (continued)

- $r(t, T)$: the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period. In other words, the $(T - t)$-period spot rate at time $t$.
  - The long rate is defined as $r(t, \infty)$.
- $F(t, T, M)$: the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$.

[Meriwether] scoring especially high marks in mathematics an indispensable subject for a bond trader.

Standard Notations (concluded)

\( f(t, T, L) \): the \( L \)-period forward rate at time \( T \) implied at time \( t \) stated on a per-period basis and compounded once per period.

\( f(t, T) \): the one-period or instantaneous forward rate at time \( T \) as seen at time \( t \) stated on a per period basis and compounded once per period.

- It is \( f(t, T, 1) \) in the discrete-time model and \( f(t, T, dt) \) in the continuous-time model.
- Note that \( f(t, t) \) equals the short rate \( r(t) \).

Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:
  
  \[
  F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \tag{87}
  \]
  
  - The forward price equals the future value at time \( T \) of the underlying asset (see text for proof).
  
- Equation \( (87) \) holds whether the model is discrete-time or continuous-time, and it implies
  
  \[
  F(t, T, M) = F(t, T, S) F(t, S, M), \quad T \leq S \leq M.
  \]

Fundamental Relations

- The price of a zero-coupon bond equals

  \[
  P(t, T) = \begin{cases} 
  (1 + r(t, T))^{(T - t)} & \text{in discrete time}, \\
  e^{r(t, T)(T - t)} & \text{in continuous time}, 
  \end{cases}
  \]

- \( r(t, T) \) as a function of \( T \) defines the spot rate curve at time \( t \).

- By definition,

  \[
  f(t, t) = \begin{cases} 
  r(t, t + 1) & \text{in discrete time}, \\
  r(t, t) & \text{in continuous time}, 
  \end{cases}
  \]

Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

  \[
  f(t, T, L) = \left( \frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1 \tag{88}
  \]

  in discrete time.

  - \( f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L)} - 1 \right) \) is the analog to Eq. \( (88) \) under simple compounding.
Fundamental Relations (continued)

- In continuous time,
  \[ f(t, T, L) = \frac{\ln F(t, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L} \]
  \[ \text{(89)} \]
  by Eq. (87) on p. 772,

- Furthermore,
  \[ f(t, T, \Delta t) = \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \to \frac{-\partial \ln P(t, T)}{\partial T} \]
  \[ = -\frac{\partial P(t, T)/\partial T}{P(t, T)}. \]

Fundamental Relations (concluded)

- The discrete analog to Eq. (91) is
  \[ P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1))(1 + f(t, T - 1))}. \]
  \[ \text{(92)} \]

- The short rate and the market discount function are related by
  \[ r(t) = -\frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}. \]
  - This can be verified with Eq. (90) on p. 775 and the observation that \( P(t, t) = 1 \) and \( r(t) = f(t, t) \).

Fundamental Relations (continued)

- So
  \[ f(t, T) \equiv \lim_{\Delta t \to 0} f(t, T, \Delta t) = -\frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T. \]
  \[ \text{(90)} \]

- Because Eq. (90) is equivalent to
  \[ P(t, T) = e^{\int_t^T f(t, s) ds}, \]
  \[ \text{(91)} \]
  the spot rate curve is
  \[ r(t, T) = \frac{1}{T-t} \int_t^T f(t, s) ds. \]

Risk-Neutral Pricing

- Under the local expectations theory, the expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all \( t + 1 < T \),
    \[ \mathbb{E}_t \left[ \frac{P(t + 1, T)}{P(t, T)} \right] = 1 + r(t). \]
    \[ \text{(93)} \]
  - Relation (93) in fact follows from the risk-neutral valuation principle, Theorem 18 (p. 428).
Risk-Neutral Pricing (continued)
- The local expectations theory is thus a consequence of the existence of a risk-neutral probability $\pi$.
- Rewrite Eq. (93) as
  $$\frac{E_\pi^T [P(t + 1, T)]}{1 + r(t)} = P(t, T).$$
- It says the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate.

Risk-Neutral Pricing (concluded)
- Equation (93) on p. 777 can also be expressed as
  $$E_\pi^T [P(t + 1, T)] = F(t, t + 1, T).$$
- Hence the forward price for the next period is an unbiased estimator of the expected bond price.

Risk-Neutral Pricing (continued)
- Apply the above equality iteratively to obtain
  $$P(t, T) = E_\pi^T \left[ \frac{P(t + 1, T)}{1 + r(t)} \right]$$
  $$= E_\pi^T \left[ \frac{E_\pi^{t+1} [P(t + 2, T)]}{(1 + r(t))(1 + r(t + 1))} \right]$$
  $$= E_\pi^T \left[ \frac{1}{(1 + r(t))(1 + r(t + 1)) \cdots (1 + r(T - 1))} \right].$$

Continuous-Time Risk-Neutral Pricing
- In continuous time, the local expectations theory implies
  $$P(t, T) = E_t^T \left[ e^{\int_t^T r(s) \, ds} \right], \quad t < T.$$  \hspace{1cm} (95)
- Note that $e^{\int_t^T r(s) \, ds}$ is the bank account process, which denotes the rolled-over money market account.
- When the local expectations theory holds, riskless arbitrage opportunities are impossible.