Trading and the Ito Integral

- Consider an Ito process \( dS_t = \mu_t dt + \sigma_t dW_t \).
  - \( S_t \) is the vector of security prices at time \( t \).
- Let \( \phi_t \) be a trading strategy denoting the quantity of each type of security held at time \( t \).
- Hence the stochastic process \( \phi_t S_t \) is the value of the portfolio \( \phi_t \) at time \( t \).
- \( \phi_t dS_t \equiv \phi_t (\mu_t dt + \sigma_t dW_t) \) represents the change in the value from security price changes occurring at time \( t \).

Ito’s Lemma

A smooth function of an Ito process is itself an Ito process.

**Theorem 20** Suppose \( f : R \rightarrow R \) is twice continuously differentiable and \( dX = a_t dt + b_t dW \). Then \( f(X) \) is the Ito process,

\[
\begin{align*}
  f(X_t) &= f(X_0) + \int_0^t f'(X_s) a_s ds + \int_0^t f'(X_s) b_s dW_s \\
  &+ \frac{1}{2} \int_0^t f''(X_s) b_s^2 ds
\end{align*}
\]

for \( t \geq 0 \).

Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

\[
G_T(\phi) = \int_0^T \phi_t dS_t = \int_0^T \phi_t \mu_t dt + \int_0^T \phi_t \sigma_t dW_t,
\]

measures the gains realized by the trading strategy over the period \([0, T]\).
- A strategy is self-financing if

\[
\phi_t S_t = \phi_0 S_0 + G_t(\phi)
\]

for all \( 0 \leq t < T \).
  - The investment at any time equals the initial investment plus the total capital gains.

Ito’s Lemma (continued)

- In differential form, Ito’s lemma becomes

\[
df(X) = f'(X) a_t dt + f'(X) b_t dW_t + \frac{1}{2} f''(X) b_t^2 dt. \tag{51}
\]

- Compared with calculus, the interesting part is the third term on the right-hand side.
- A convenient formulation of Ito’s lemma is

\[
df(X) = f'(X) dX + \frac{1}{2} f''(X)(dX)^2. \tag{52}
\]
Ito's Lemma (continued)

- We are supposed to multiply out $(dX)^2 = (a \, dt + b \, dW)^2$ symbolically according to

<table>
<thead>
<tr>
<th>$\times$</th>
<th>$dW$</th>
<th>$dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dW$</td>
<td>$dt$</td>
<td>0</td>
</tr>
<tr>
<td>$dt$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- The $(dW)^2 = dt$ entry is justified by a known result.
- This form is easy to remember because of its similarity to the Taylor expansion.

Ito's Lemma (concluded)

- The multiplication table for Theorem 21 is

<table>
<thead>
<tr>
<th>$\times$</th>
<th>$dW_i$</th>
<th>$dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dW_k$</td>
<td>$\delta_{ik} , dt$</td>
<td>0</td>
</tr>
<tr>
<td>$dt$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

in which

$$\delta_{ik} = \begin{cases} 
1 & \text{if } i = k, \\
0 & \text{otherwise.}
\end{cases}$$

Theorem 21 (Higher-Dimensional Ito's Lemma) Let $W_1, W_2, \ldots, W_n$ be independent Wiener processes and $X = (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable and $X_i$ is an Ito process with $dX_i = a_i \, dt + \sum_{j=1}^{n} b_{ij} \, dW_j$. Then $df(X)$ is an Ito process with the differential,

$$df(X) = \sum_{i=1}^{m} f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X) \, dX_i \, dX_k,$$

where $f_i \equiv \partial f / \partial x_i$ and $f_{ik} \equiv \partial^2 f / \partial x_i \partial x_k$.

Geometric Brownian Motion

- Consider the geometric Brownian motion process $Y(t) \equiv e^{X(t)}$
  - $X(t)$ is a $(\mu, \sigma)$ Brownian motion,
- As $\partial Y / \partial = Y$ and $\partial^2 Y / \partial X^2 = Y$, Ito's formula (51) on p. 467 implies

$$\frac{dY}{Y} = (\mu + \sigma^2/2) \, dt + \sigma \, dW,$$

- The annualized instantaneous rate of return is $\mu + \sigma^2/2$ not $\mu$. 
Ornstein-Uhlenbeck Process

- The Ornstein-Uhlenbeck process:
  \[ dX = -\kappa X \, dt + \sigma dW, \]
  where \( \kappa, \sigma \geq 0. \)

- It is known that
  
  \[
  E[X(t)] = e^{\kappa t} E[X_0], \\
  \text{Var}[X(t)] = \frac{\sigma^2}{2\kappa} \left( 1 - e^{-2\kappa(t - t_0)} \right) - e^{-2\kappa(t - t_0)} \text{Var}[X_0], \\
  C_{ov}[X(s), X(t)] = \frac{\sigma^2}{2\kappa} \left[ 1 - e^{-2\kappa(s - t_0)} \right] \\
  + e^{-2\kappa(t - s)} \text{Var}[X_0],
  \]
  for \( t_0 \leq s \leq t \) and \( X(t_0) = x_0. \)

Ornstein-Uhlenbeck Process (continued)

- \( X(t) \) is normally distributed if \( x_0 \) is a constant or normally distributed.

- \( X \) is said to be a normal process.

- \( E[x_0] = x_0 \) and \( \text{Var}[x_0] = 0 \) if \( x_0 \) is a constant.

- The Ornstein-Uhlenbeck process has the following mean reversion property.
  - When \( X > 0 \), \( X \) is pulled \( X \) toward zero.
  - When \( X < 0 \), it is pulled toward zero again.

Ornstein-Uhlenbeck Process (continued)

- Another version:
  \[ dX = \kappa(\mu - X) \, dt + \sigma dW, \]
  where \( \sigma \geq 0. \)

- Given \( X(t_0) = x_0 \); a constant, it is known that
  
  \[
  E[X(t)] = \mu + (x_0 - \mu) e^{-\kappa(t - t_0)}, \\
  \text{Var}[X(t)] = \frac{\sigma^2}{2\kappa} \left[ 1 - e^{-2\kappa(t - t_0)} \right],
  \]
  for \( t_0 \leq t. \)

Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly \( \mu \) and \( \sigma/\sqrt{2\kappa} \), respectively.

- For large \( t \), the probability of \( X < 0 \) is extremely unlikely in any finite time interval when \( \mu > 0 \) is large relative to \( \sigma/\sqrt{2\kappa} \) (say \( \mu > 4\sigma/\sqrt{2\kappa} \)).

- The process is mean-reverting.
  - \( X \) tends to move toward \( \mu. \)
  - Useful for modeling term structure, stock price volatility, and stock price return.
Interest Rate Models

- Suppose the short rate \( r \) follows process 
  \[ dr = \mu(r, t) \, dt + \sigma(r, t) \, dW. \]
- Let \( P(r, t, T) \) denote the price at time \( t \) of a 
  zero-coupon bond that pays one dollar at time \( T \).
- Write its dynamics as 
  \[ \frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW. \]
  - The expected instantaneous rate of return on a 
    \((T - t)\)-year zero-coupon bond is \( \mu_p \).
  - The instantaneous variance is \( \sigma_p^2 \).

\(^a\)Merton (1970).

Interest Rate Models (concluded)

- Hence, 
  \[ - \frac{\partial P}{\partial t} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} = P \mu_p, \quad (54) \]
  \[ \sigma(r, t) \frac{\partial P}{\partial r} = P \sigma_p. \]
- Models with the short rate as the only explanatory 
  variable are called short rate models.

Interest Rate Models (continued)

- Surely \( P(r, T, T) = 1 \) for any \( T \).
- By Itô's lemma (Theorem 21 on p. 469), 
  \[ dP = \left[ - \frac{\partial P}{\partial t} + \frac{\mu(r, t)}{2} \frac{\partial^2 P}{\partial r^2} + \frac{\sigma(r, t)^2}{2} \right] dt \]
  \[ + \sigma(r, t) \frac{\partial P}{\partial r} \, dW. \]

The Merton Model

- Assume the local expectations theory, which means \( \mu_p \) 
  equals the prevailing short rate \( r(t) \) for all \( T \).
- Assume further that \( \mu \) and \( \sigma \) are constants,
- Then the partial differential equations (54) yield 
  \[ P(r, t, T) = e^{- \int_t^T r(s) \, ds} \frac{r(t)^2}{2} \frac{\partial^2 P}{\partial r^2} \]
  \[ + \sigma(r, t)^2 \frac{\partial P}{\partial r} \, dW. \]
- The dynamics of \( P \) is \( dP/P = r \, dt - \sigma(T - t) \, dW. \)
- Now, \( P \) has no upper limits as \( T \) becomes large, which 
  does not square with the reality.
Continuous-Time Derivatives Pricing

Toward the Black-Scholes Differential Equation
- The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation.
- The key step is recognizing that the same random process drives both securities.
- As their prices are perfectly correlated, we figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative.
- The removal of uncertainty forces the portfolio’s return to be the riskless rate.

Assumptions
- The stock price follows $dS = \mu S \, dt + \sigma S \, dW$.
- There are no dividends.
- Trading is continuous, and short selling is allowed.
- There are no transactions costs or taxes.
- All securities are infinitely divisible.
- The term structure of riskless rates is flat at $r$.
- There is unlimited riskless borrowing and lending.
- $t$ is the current time, $T$ is the expiration time, and $\tau \equiv T - t$.

I have hardly met a mathematician who was capable of reasoning. Plato (428 B.C. 347 B.C.)
Black-Scholes Differential Equation

- Let $C$ be the price of a derivative on $S$.
- From Ito’s lemma (p. 469),
  \[ dC = \left( \mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW. \]
  - The same $W$ drives both $C$ and $S$.
- Short one derivative and long $\partial C/\partial S$ shares of stock (call it $\Pi$).
- By construction,
  \[ \Pi = -C + S(\partial C/\partial S). \]

Black-Scholes Differential Equation (continued)

- The change in the value of the portfolio at time $dt$ is
  \[ d\Pi = -dC + \frac{\partial C}{\partial S} dS. \]
- Substitute the formulas for $dC$ and $dS$ into the partial differential equation to yield
  \[ d\Pi = \left( -\frac{\partial C}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt. \]
- As this equation does not involve $dW$, the portfolio is riskless during $dt$ time: $d\Pi = r\Pi dt$.

Black-Scholes Differential Equation (concluded)

- So
  \[ \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt = r \left( C - S \frac{\partial C}{\partial S} \right) dt. \]
- Equate the terms to finally obtain
  \[ \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC. \]
- When there is a dividend yield $q$,
  \[ \frac{\partial C}{\partial t} + (r-q) S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC. \]

Rephrase

- The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,
  \[ \Theta + rS \Delta + \frac{1}{2} \sigma^2 S^2 \Gamma = rC. \]
- Identity (55) leads to an alternative way of computing $\Theta$ numerically from $\Delta$ and $\Gamma$.
- When a portfolio is delta-neutral,
  \[ \Theta + \frac{1}{2} \sigma^2 S^2 \Gamma = rC. \]
  - A definite relation thus exists between $\Gamma$ and $\Theta$. 
Exchange Options

- A correlation option has value dependent on multiple assets.
- An exchange option is a correlation option.
- It gives the holder the right to exchange one asset for another.
- Its value at expiration is thus

\[\max(S_2(T) - S_1(T), 0),\]

where \( S_1(T) \) and \( S_2(T) \) denote the prices of the two assets at expiration.

\textsuperscript{a}Majrabe (1978).

Pricing of Exchange Options

- Assume that the two underlying assets do not pay dividends and that their prices follow

\[
\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dW_1, \\
\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dW_2,
\]

where \( \rho \) is the correlation between \( dW_1 \) and \( dW_2 \).

Exchange Options (concluded)

- The payoff implies two ways of looking at the option.
  - It is a call on asset 2 with a strike price equal to the future price of asset 1.
  - It is a put on asset 1 with a strike price equal to the future value of asset 2.

Pricing of Exchange Options (concluded)

- The option value at time \( t \) is

\[
V(S_1, S_2, t) = S_2 N(x) - S_1 N(x - \sigma \sqrt{T - t}),
\]

where

\[
x = \frac{\ln(S_2/S_1) + (\sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \\
\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2,
\]

where \( \sigma^2 \) is given by \( \sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \).

- This is called Majrabe's formula.
Derivation of Margrabe's Formula

- Observe first that $V(x, y, t)$ is homogeneous of degree one in $x$ and $y$.
  - That is, $V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t)$.
  - An exchange option based on $\lambda$ times the prices of the two assets is thus equal in value to $\lambda$ original exchange options.
  - Intuitively, this is true because of
    \[ \max(\lambda S_2(T) - \lambda S_1(T), 0) = \lambda \max(S_2(T) - S_1(T), 0) \]
    and the perfect market assumption.

Derivation of Margrabe's Formula (concluded)

- The option to exchange asset 1 for asset 2 is a call on asset 2 with a strike price equal to unity and the interest rate equal to zero.
- So the Black-Scholes formula applies:
  \[
  \frac{V(S_1, S_2, t)}{S_1} = V(1, S, t) = SN(x) - 1 \times e^{-\sigma \sqrt{T-t}} N(x - \sigma \sqrt{T-t}),
  \]
  where
  \[
  x \equiv \frac{\ln(S_1/S_2) + (0 + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = \frac{\ln(S_2/S_1) + (0 + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.
  \]

Margrabe's Formula with Dividends

- Margrabe's formula is not much more complicated if $S_i$ pays out a continuous dividend yield of $q_i$, $i = 1, 2$.
- Simply replace each occurrence of $S_i$ with $S_i e^{-q_i(T-t)}$ to obtain
  \[
  V(S_1, S_2, t) = S_2 e^{-q_2(T-t)} N(x) - S_1 e^{-q_1(T-t)} N(x - \sigma \sqrt{T-t}),
  \]
  where
  \[
  x \equiv \frac{\ln(S_2/S_1) + (q_1 - q_2 + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}},
  \]
  \[
  \sigma^2 \equiv \sigma_1^2 - 2p\sigma_1\sigma_2 + \sigma_2^2.
  \]
Options on Foreign Currencies and Assets

- Correlation options involving foreign currencies and assets can be analyzed either take place in the domestic market or the foreign market before being converted back into the domestic currency.
- In the following, \( S(t) \) denotes the spot exchange rate in terms of the domestic value of one unit of foreign currency.
- We knew from p. 328 that foreign currency is analogous to a stock paying a continuous dividend yield equal to the foreign riskless interest rate \( r_f \) in foreign currency.

Foreign Equity Options

- From Eq. (26) on p. 268, a European option on the foreign asset \( G_t \) with the terminal payoff \( S(T) \times \max(G_t(T) - X_t, 0) \) is worth

\[
C_t = G_t e^{-r_f \tau} N(x) - X_t e^{-r_f \tau} N(x - \sigma_f \sqrt{\tau})
\]

in foreign currency.
- Above,

\[
x = \ln(G_t / X_t) + (r_f - q_f + \sigma_f^2 / 2) \tau / \sigma_f \sqrt{\tau}.
\]
- \( X_t \) is the strike price in foreign currency.

Options on Foreign Currencies and Assets (concluded)

- So \( S(t) \) follows the geometric Brownian motion process,

\[
\frac{dS}{S} = (r_f - r_d) dt + \sigma_s dW_s(t),
\]

in a risk-neutral economy.
- The foreign asset will be assumed to pay a continuous dividend yield of \( q_f \), and its price follows

\[
\frac{dG_t}{G_t} = (\mu_f - q_f) dt + \sigma_f dW_f(t)
\]

in foreign currency.
- \( \rho \) is the correlation between \( dW_s \) and \( dW_f \).

Foreign Equity Options (concluded)

- Similarly, a European option on the foreign asset \( G_t \) with the terminal payoff \( S(T) \times \max(X_t - G_t(T), 0) \) is worth

\[
P_t = X_t e^{-r_f \tau} N(-x + \sigma_f \sqrt{\tau}) - G_t e^{-r_f \tau} N(-x)
\]

in foreign currency.
- They will fetch \( SC_t \) and \( SP_t \), respectively, in domestic currency.
- These options are called foreign equity options struck in foreign currency.
Foreign Domestic Options

- Foreign equity options fundamentally involve values in the foreign currency.
- But although a foreign equity call may allow the holder to participate in a foreign market rally, the profits can be wiped out if the foreign currency depreciates against the domestic currency.
- What is really needed is a call in domestic currency with a payoff of $\max(S(T) G_t(T) - X, 0)$.
  - For foreign equity options, the strike price in domestic currency is the uncertain $S(T) X_t$.
- This is called a foreign domestic option.

Pricing of Foreign Domestic Options (concluded)

- The domestic price is therefore
  \[ C = S G_t e^{-q T} N(x) - X e^{-r T} N(x - \sigma \sqrt{T}). \]
- Similarly, a put has a price of
  \[ P = X e^{-r T} N(-x + \sigma \sqrt{T}) - S G_t e^{-q T} N(-x). \]

Pricing of Foreign Domestic Options

- To foreign investors, this call is an option to exchange $X$ units of domestic currency (foreign currency to them) for one share of foreign asset (domestic asset to them).
- It is an exchange option, that is,
- By Eq. (57) on p. 495, its price in foreign currency equals
  \[ G_t e^{-q T} N(x) = \frac{X}{S} e^{-r T} N(x - \sigma \sqrt{T}), \]
  \[ x \equiv \frac{\ln(G_t S/X) + (r - q_t + \sigma^2/2) \tau}{\sigma \sqrt{T}}, \]
  \[ \sigma^2 \equiv \sigma_e^2 + 2 \rho \sigma_e \sigma_t + \sigma_t^2. \]

Quanto Options

- Consider a call with a terminal payoff $\tilde{S} \times \max(G_t(T) - X, 0)$ in domestic currency, where $\tilde{S}$ is a constant.
- This amounts to fixing the exchange rate to $\tilde{S}$.
  - For instance, a call on the Nickel 225 futures, if it existed, fits this framework with $\tilde{S} = 5$ and $G_t$ denoting the futures price.
- A guaranteed exchange rate option is called a quanto option or simply a quanto.
Quanto Options (continued)

- The process $U \equiv \tilde{S}_t$ in a risk-neutral economy follows

$$\frac{dU}{U} = (r_t - q_t - \rho \sigma_u \sigma_t) dt + \sigma_t dW$$  \hspace{1cm} (58)

in domestic currency.

- Hence, it can be treated as a stock paying a continuous dividend yield of $q \equiv r - r_t + q_t + \rho \sigma_u \sigma_t$.

- Apply Eq. (26) on p. 268 to obtain

$$C = \tilde{S} \left( G_t e^{q_t \sqrt{\tau}} N(x) - X_t e^{r \sqrt{\tau}} N(x - \sigma_t \sqrt{\tau}) \right)$$

$$P = \tilde{S} \left( G_t e^{-q_t \sqrt{\tau}} N(-x + \sigma_t \sqrt{\tau}) - G_t e^{-r \sqrt{\tau}} N(-x) \right)$$

where $x \equiv \frac{\ln(G_t/X_t) + (r - q + \sigma_t^2/2) \tau}{\sigma_t \sqrt{\tau}}$.

---

General Derivatives Pricing

- In general the underlying asset $S$ may not be traded.

  - Interest rate, for instance, is not a traded security.

- Let $S$ follow the Ito process $dS/S = \mu dt + \sigma dW$, where $\mu$ and $\sigma$ may depend only on $S$ and $t$.

- Let $f_1(S,t)$ and $f_2(S,t)$ be the prices of two derivatives with dynamics $df_i/f_i = \mu_i dt + \sigma_i dW$, $i = 1, 2$.

  - They share the same Wiener process as $S$.

---

Quanto Options (concluded)

- In general, a quanto derivative has nominal payments in the foreign currency which are converted into the domestic currency at a fixed exchange rate.

- A cross-rate swap, for example, is like a currency swap except that the foreign currency payments are converted into the domestic currency at a fixed exchange rate.

- Quanto derivatives form a rapidly growing segment of international financial markets.

---

General Derivatives Pricing (continued)

- A portfolio consisting of $\sigma_2 f_2$ units of the first derivative and $-\sigma_1 f_1$ units of the second derivative is instantaneously riskless:

$$\sigma_2 f_2 df_1 - \sigma_1 f_1 df_2$$

$$= \sigma_2 f_2 (\mu_1 dt + \sigma_1 dW) - \sigma_1 f_1 (\mu_2 dt + \sigma_2 dW)$$

$$= (\sigma_2 f_2 \mu_1 - \sigma_1 f_1 \mu_2) dt.$$  

- Therefore,

$$\sigma_2 \mu_1 - \sigma_1 \mu_2 = r(\sigma_2 f_2 - \sigma_1 f_1) dt,$$

or $\sigma_2 \mu_1 - \sigma_1 \mu_2 = r(\sigma_2 - \sigma_1)$.  

General Derivatives Pricing (continued)

- After rearranging the terms,
  \[ \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda \] for some \( \lambda \).

- A derivative whose value depends only on \( S \) and \( t \) and which follows the Ito process \( df/f = \mu dt + \sigma dW \) must thus satisfy
  \[ \frac{\mu - r}{\sigma} = \lambda \] or \( \mu = r + \lambda \sigma \). (59)

- We call \( \lambda \) the market price of risk, which is independent of the specifics of the derivative.

General Derivatives Pricing (continued)

- Ito's lemma can be used to derive the formulas for \( \mu \) and \( \sigma \):
  \[ \mu = \frac{1}{f} \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) \]
  \[ \sigma = \frac{\sigma S}{f} \frac{\partial f}{\partial S} \]

- Substitute the above into Eq. (39) to obtain
  \[ \frac{\partial f}{\partial t} + (\mu - \lambda \sigma) S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = \lambda f \]. (60)

Hedging
Delta Hedge

- The delta (hedge ratio) of a derivative $f$ is defined as $\Delta = \partial f / \partial S$.
- Thus $\Delta f \approx \Delta \times DS$ for relatively small changes in the stock price, $DS$.
- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.
- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.

Delta Hedge (concluded)

- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.
- This was the gist of the Black-Scholes-Merton argument.

Implementing Delta Hedge

- We want to hedge $N$ short derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus $B$ borrowed dollars such that
  $$-N \times f + N \times \Delta \times S - B = 0,$$
- At next rebalancing point when the delta is $\Delta'$, buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$.
- Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.
Example

- A hedger is short 10,000 European calls.
- \( \sigma = 30\% \) and \( r = 6\% \).
- This call’s expiration is four weeks away, its strike price is \$50, and each call has a current value of \( f = 1.76791 \).
- As an option covers 100 shares of stock, \( N = 1,000,000 \).
- The trader adjusts the portfolio weekly.
- The calls are replicated well if the cumulative cost of trading stock is close to the call premium’s FV.

Example (continued)

- As \( \Delta = 0.538560 \), \( N \times \Delta = 538,560 \) shares are purchased for a total cost of \( 538,560 \times 50 = 26,928,000 \) dollars to make the portfolio delta-neutral.
- The trader finances the purchase by borrowing
  \[ B = N \times \Delta \times S - N \times f = 25,160,090 \]
dollars net.
- The portfolio has zero net value now.

Example

- At 3 weeks to expiration, the stock price rises to \$51.
- The new call value is \( f' = 2.10580 \).
- So the portfolio is worth
  \[ -N \times f' + 538,560 \times 51 - Be^{0.06/52} = 171,622 \]
before rebalancing.
Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as $171,622 can be withdrawn.
- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.
- In fact, the tracking error is positive about 68% of the time even though its expected value is essentially zero.\(^a\)
- It is furthermore proportional to vega.

\(^a\)Boyle and Emanuel (1980).

Example (continued)

- The cumulative cost is
  \[ 26,928,000 \times e^{0.06/52} + 5,191,545 = 32,150,634. \]
- The net borrowed amount is
  \[ B' = 640,355 \times 51 - N \times f' = 30,552,305. \]
- The portfolio is again delta-neutral with zero value.

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Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta \( \Delta' = 0.640355 \), the trader buys \( N \times (\Delta' - \Delta) = 101,795 \) shares for $5,191,545.
- The number of shares is increased to \( N \times \Delta' = 640,355 \).

<table>
<thead>
<tr>
<th>Option</th>
<th>Delta</th>
<th>Delta</th>
<th>Change in Delta</th>
<th>No. Shares</th>
<th>Cost of Shares</th>
<th>Cumulative Cost</th>
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<tbody>
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<tr>
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<td>0.75</td>
<td>250,000</td>
<td>15,000</td>
<td>31,524,653</td>
</tr>
</tbody>
</table>

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).
Example (concluded)

- At expiration, the trader has 1,000,000 shares,
- They are exercised against by the in-the-money calls for $50,000,000.
- The trader is left with an obligation of
  \[ 51,524,853 - 50,000,000 = 1,524,853, \]
  which represents the replication cost,
- Compared with the FV of the call premium,
  \[ 1,767,910 \times e^{0.06 \times 4/52} = 1,776,088, \]
  the net gain is \[ 1,776,088 - 1,524,853 = 251,235, \]