Diagonal Traversal of the Tree (continued)

Two properties of the propagation of early exercise nodes (E) and non-early exercise nodes (C) during backward induction are:

1. A node is an early exercise node if both its successor nodes are exercised early.
   - A terminal node that is in-the-money is considered an early exercise node.

2. If a node is a non-early exercise node, then all the earlier nodes at the same horizontal level are also non-early exercise nodes.
   - Here we assume $ud = 1$.

Diagonal Traversal of the Tree (continued)

- Nothing is achieved if the whole tree needs to be explored.
- We need a stopping rule.
- The process stops when a diagonal $D$ consisting entirely of non-early exercise nodes has been encountered.
  - By Rule 2, all early exercise nodes have been found.

Diagonal Traversal of the Tree (continued)

- When the algorithm finds an early exercise node in traversing a diagonal, it can stop immediately and move on to the next diagonal.
  - By Rule 1 and the sequence by which the nodes on the diagonals are traversed, the rest of the nodes on the current diagonal must all be early exercise nodes.
  - They are hence computable on the fly when needed.
Diagonal Traversal of the Tree (continued)

- Also by Rule 1, the traversal can start from the zero-valued terminal node just above the strike price.
- The upper triangle above the strike price can be skipped since its nodes are all zero valued.

Diagonal Traversal of the Tree (continued)

- It remains to calculate the option value.
- It is the weighted sum of the discounted option values of the nodes on $D$.
  - How does the payoff influence the root?
  - We cannot go from the root to a node at which the option will be exercised without passing through $D$.
- The weight is the probability that the stock price hits the diagonal for the first time at that node.
Diagonal Traversal of the Tree (concluded)

- For a node on $D$ which is the result of $i$ up moves and $j$ down moves, the said probability is $(i+j-1) p^i (1-p)^j$.
- A valid path must pass through the node which is the result of $i$ up moves and $j-1$ down moves.
- Call the option value on this node $P_i$.
- The desired option value then equals

$$
\sum_{i=0}^{\alpha} \binom{i+j-1}{i} p^i (1-p)^j P e^{(i+j) r \Delta \tau},
$$

Sensitivity Analysis of Options

The Analysis

- Since each node on $D$ has been evaluated by that time, this part of the computation consumes $O(n)$ time.
- The space requirement is also linear in $n$ since only the diagonal has to be allocated space.
- This idea can save computation time when $D$ does not take long to find.
- Proof of Rule 1 and 2 is in the text.

Sensitivity Measures ("The Greeks")

- Understanding how the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- We now ask the same questions of options.
- Let $x = \frac{\ln(S/X)+(r+\sigma^2/2)\tau}{\sigma \sqrt{\tau}}$ (recall p. 254).
- Note that

$$
N'(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} > 0,
$$

the density function of standard normal distribution.
Delta

- Defined as \( \Delta \equiv \partial f / \partial S \).
  - \( f \) is the price of the derivative.
  - \( S \) is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.
- The delta used in the BOPM is the discrete analog.

Delta (concluded)

- The delta of a European call on a non-dividend-paying stock equals
  \[
  \frac{\partial C}{\partial S} = N(x) > 0.
  \]
- The delta of a European put equals
  \[
  \frac{\partial P}{\partial S} = N(x) - 1 < 0.
  \]
- The delta of a long stock is 1.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
- A delta-neutral portfolio is immune to small price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and \(-\Delta\) shares of stock is delta-neutral.
  - Short \( \Delta \) shares of stock to hedge a long call.
- Hedge a position in a security with a delta of \( \Delta_1 \) by shorting \( \Delta_1 / \Delta_2 \) units of a security with a delta of \( \Delta_2 \).

Theta (Time Decay)

- Defined as the rate of change of a security's value with respect to time, or \( \Theta \equiv -\partial f / \partial \tau \).
- For a European call on a non-dividend-paying stock,
  \[
  \Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0.
  \]
  - The call loses value with the passage of time.
- For a European put,
  \[
  \Theta = -\frac{SN'(x)\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}).
  \]
  - Can be negative or positive.
Gamma
- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2 \Pi / \partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- A portfolio with a high gamma needs in practice be rebalanced more often to maintain delta neutrality.
- Delta $\sim$ duration; gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is $N'(x)/(S\sigma \sqrt{T}) > 0$.

Rho
- Defined as the rate of change in its value with respect to interest rates $\rho \equiv \partial \Pi / \partial r$.
- The rho of a European call on a non-dividend-paying stock is $Xe^{-rT}N(x - \sigma \sqrt{T}) > 0$.
- The rho of a European put on a non-dividend-paying stock is $-Xe^{-rT}N(-x + \sigma \sqrt{T}) < 0$.

Vega\(^a\) (Lambda, Kappa, Sigma)
- Defined as the rate of change of its value with respect to the volatility of the underlying asset $\Lambda \equiv \partial \Pi / \partial \sigma$.
- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{T}N'(x) > 0$.
  - Higher volatility increases option value.

\(^a\)Vega is not Greek.

Numerical Greeks
- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference, $f(S + \Delta S) - f(S - \Delta S) / 2\Delta S$.
- The computation time roughly doubles that for evaluating the derivative security itself.
An Alternative Numerical Delta

- Use intermediate results of the binomial tree algorithm,
- When the algorithm reaches the end of the first period, \( f_u \) and \( f_d \) are computed.
- These values correspond to derivative values at stock prices \( S_u \) and \( S_d \), respectively.
- Delta is approximated by \( \frac{f_u - f_d}{S_u - S_d} \).
- Almost zero extra computational effort.

\(^a\)Pésser and Vorst (1994).

---

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives
  \[
  \frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.
  \]
- It does not work (see text).
- Why did the binomial tree version work?

---

Numerical Gamma

- At the stock price \( (S_u + S_d)/2 \), delta is approximately \( (f_{uu} - f_{ud})/(S_u - S_d) \).
- At the stock price \( (S_d + S_dd)/2 \), delta is approximately \( (f_{ud} - f_{dd})/(S_d - S_dd) \).
- Gamma is the rate of change in deltas between \( (S_u + S_d)/2 \) and \( (S_d + S_dd)/2 \), that is,
  \[
  \frac{f_{uu} f_{dd} - f_{ud} f_{dd}}{(S_u - S_d)/2}.
  \]

---

Other Numerical Greeks

- The theta can be computed as
  \[
  \frac{f_{ud} - f}{2(\pi/n)}.
  \]
  - In fact, the theta of a European option will be shown to be computable from delta and gamma.
- For vega and rho, there is no alternative but to run the binomial tree algorithm twice.
Pricing Corporate Securities

- Interpret the underlying asset interpreted as the total value of the firm.
- The option pricing methodology can be applied to pricing corporate securities.
- Assume:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

---

Extensions of Options Theory

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - $n$ shares of its own common stock, $S$.
  - Zero-coupon bonds with an aggregate par value of $X$.
- What is the value of the bonds, $B$?
- What is the value of the XYZ.com stock?
Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, if the total value of the firm $V^*$ is less than the bondholders' claim $X$, the firm declares bankruptcy and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain $X$ and the stockholders $V^* - X$.

<table>
<thead>
<tr>
<th>$V^* \leq X$</th>
<th>$V^* &gt; X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>$V^*$</td>
</tr>
<tr>
<td>Stock</td>
<td>0</td>
</tr>
</tbody>
</table>

Risky Zero-Coupon Bonds and Stock (continued)

- Thus $nS = C$ and $B = V - C$.
- Knowing $C$ amounts to knowing how the value of the firm is divided between the stockholders and the bondholders.
- Whatever the value of $C$, the total value of the stock and bonds at maturity remains $V^*$.
- The relative size of debt and equity is irrelevant to the firm's current value $V$.

Risky Zero-Coupon Bonds and Stock (continued)

- The stock is a call on the total value of the firm with a strike price of $X$ and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call on the total value of the firm.
- Let $V$ stand for the total value of the firm.
- Let $C$ stand for the call.

Risky Zero Coupon Bonds and Stock (concluded)

- From Theorem 12 (p. 254) and the put-call parity,
  $$ nS = VN(x) = Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), $$
  $$ B = VN(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). $$
  - where
  $$ x = \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}. $$
- The continuously compounded yield to maturity of the firm's bond is
  $$ \frac{\ln(X/B)}{\tau}. $$
A Numerical Example

- XYZ.com’s assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck’s market value per share is $44.5.
- XYZ.com’s securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay $1,000 at maturity.
- \( n = 1000, V = 44.5 \times n = 44500 \), and \( X = 30 \times n = 30000 \).

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of \( X/n = 30 \) dollars.
- Such a call is selling for $15.25.
- So XYZ.com’s stock is worth \( 15.25 \times n = 15250 \) dollars.
- The entire bond issue is worth \( B = 44500 - 15250 = 29250 \) dollars.
  - Or $975 per bond.

<table>
<thead>
<tr>
<th>Option</th>
<th>Strike</th>
<th>Exp.</th>
<th>Vol.</th>
<th>Last</th>
<th>Put</th>
<th>Vol.</th>
<th>Last</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merck</td>
<td>30</td>
<td>Jul</td>
<td>328</td>
<td>151/4</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>35</td>
<td>Jul</td>
<td>150</td>
<td>91/2</td>
<td>10</td>
<td>1/16</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Apr</td>
<td>887</td>
<td>43/4</td>
<td>136</td>
<td>1/16</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Jul</td>
<td>220</td>
<td>51/2</td>
<td>297</td>
<td>1/4</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Oct</td>
<td>58</td>
<td>6</td>
<td>10</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Apr</td>
<td>3050</td>
<td>7/8</td>
<td>100</td>
<td>1/8</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>May</td>
<td>462</td>
<td>13/8</td>
<td>50</td>
<td>13/8</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Jul</td>
<td>883</td>
<td>115/16</td>
<td>147</td>
<td>13/4</td>
<td></td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Oct</td>
<td>367</td>
<td>21/4</td>
<td>188</td>
<td>21/16</td>
<td></td>
</tr>
</tbody>
</table>

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with \( X \) par value plus \( n \) written European puts on Merck at a strike price of $30.
  - By the put-call parity.
- The difference between \( B \) and the price of the default-free bond is the value of these puts.
- The table next page shows the total market values of the XYZ.com stock and bonds under various debt amounts \( X \).
A Numerical Example (continued)

- Suppose the promised payment to bondholders is $45,000.
- Then the relevant option is the July call with a strike price of \( 45000/n = 45 \) dollars.
- Since that option is selling for $115/16, the market value of the XYZ.com stock is \( (1 + 15/16) \times n = 1937.5 \) dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

<table>
<thead>
<tr>
<th>Promised payment to bondholders</th>
<th>Current market value of bonds</th>
<th>Current market value of stock</th>
<th>Current total value of firm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>( B )</td>
<td>( n )</td>
<td>( V )</td>
</tr>
<tr>
<td>30,000</td>
<td>29,250.0</td>
<td>15,250.0</td>
<td>44,500</td>
</tr>
<tr>
<td>35,000</td>
<td>35,000.0</td>
<td>9,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>40,000</td>
<td>39,000.0</td>
<td>5,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>45,000</td>
<td>42,562.5</td>
<td>1,937.5</td>
<td>44,500</td>
</tr>
</tbody>
</table>

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option’s terms cannot be changed after issuance.
- But a firm can change its capital structure,
- There lies one difference between options and corporate securities.
- So parameters such volatility, dividend, and strike price are under partial control of the stockholders.

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now \( X = 45,000 \) dollars.
- The table on p. 309 says the total market value of the bonds should be $42,562.5.
- The new bondholders pay $42,562.5 \times (15/45) = 14187.5 dollars.
- The remaining stock is worth $1,937.5.
A Numerical Example (continued)

- The stockholders therefore gain
  \[ 14187.5 + 1937.5 - 15250 = 875 \] dollars.

- The original bondholders lose an equal amount,
  \[ 29250 - \frac{30}{45} \times 42562.5 = 875 \]
  \[ (28) \]

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence
  \[ (2/3) \times n \times 44.5 - 1291.67 = 28375 \] dollars.

- Hence the stockholders gain
  \[ 14833.3 + 1291.67 - 15250 \approx 875 \] dollars.

- The bondholders watch their value drop from $29,250 to $28,375, a loss of $875.

A Numerical Example (continued)

- Suppose the stockholders distribute $14,833.3 cash dividends by selling \((1/3) \times n\) Merck shares.

- They now have $14,833.3 in cash plus a call on \((2/3) \times n\) Merck shares.

- The strike price remains \(X = 30000\).

- This is equivalent to owning two-thirds of a call on \(n\) Merck shares with a total strike price of $45,000.

- The \(n\) such calls are worth \$1,937.5 (see p. 306).

- So the total market value of the XYZ.com stock is \((2/3) \times 1937.5 = 1291.67\) dollars.

Other Examples

- Subordinated debts as bull call spreads.

- Warrants as calls.

- Callable bonds as American calls with 2 strike prices.

- Convertible bonds.