Early Exercise

- Since the call will not be exercised at time one even if it is American, \( C_u \geq Su - X \) and \( C_d \geq Sd - X \).
- Therefore,
  \[
  hS + B = \frac{pC_u + (1 - p)C_d}{R} \geq \frac{[p(Su - X) + (1 - p)(Sd - X)]}{R} = S - \frac{X}{R} > S - X.
  \]
- So the call again will not be exercised at present, and
  \[
  C = hS + B = \frac{pC_u + (1 - p)C_d}{R}.
  \]

Backward Induction (continued)

- In the \( n \)-period case,
  \[
  C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j(1 - p)^{n-j} \max(0, Sw^j d^n - X)}{R^n}.
  \]
  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.
- The value of a European put is
  \[
  P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j(1 - p)^{n-j} \max(0, X - Sw^j d^n - X)}{R^n}.
  \]

Backward Induction of Zermelo (1871–1953)

- The above expression calculates \( C \) from the two successor nodes \( C_u \) and \( C_d \) and none beyond.
- The same computation happens at \( C_u \) and \( C_d \) too, as demonstrated in Eq. (23) on p. 223.
- This recursive procedure is called backward induction.
- Now, \( C \) equals
  \[
  [p^2 C_u + 2p(1 - p) C_{ud} + (1 - p)^2 C_d]/R^2 \]
  \[
  = [p^2 \max(0, Su^2 - X) + 2p(1 - p) \max(0, Sud - X) + (1 - p)^2 \max(0, Sd^2 - X)]/R^2.
  \]

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function \( D \), its value is
  \[
  e^{-\delta n} \mathbb{E}[D].
  \]
  - \( \mathbb{E}[D] \) means the expectation is taken under the risk-neutral probability.
- The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.
Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.
  - Changes in value are due entirely to capital gains.

The Binomial Option Pricing Formula (concluded)

Hence,

\[ C = \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \left( S u^j d^{n-j} - X \right) \]

\[ = S \sum_{j=0}^{n} \binom{n}{j} (pu)^j (1-p)^{n-j} \left( d^n - X \right) \sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \]

\[ = S \sum_{j=0}^{n} b(j; n, pu, e^{-r}) - X e^{-r} \sum_{j=0}^{n} b(j; n, p). \]

The Binomial Option Pricing Formula

- Let \( a \) be the minimum number of upward price moves for the call to finish in the money.
- So \( a \) is the smallest nonnegative integer such that
  \[ S u^a d^{n-a} \geq X, \]
  or
  \[ a = \left[ \frac{\ln(X/Sd^n)}{\ln(u/d)} \right]. \]

Numerical Examples

- A non-dividend-paying stock is selling for $160.
- \( u = 1.5 \) and \( d = 0.5 \).
- \( r = 18.232\% \) per period.
- Consider a European call on this stock with \( X = 150 \) and \( n = 3 \).
- The call value is $85,069 by backward induction.
- Also the PV of the expected payoff at expiration,
  \[ 390 \times 0.343 + 30 \times 0.441 + 0.189 + 0 \times 0.027 = \frac{(1.2)^3}{85,069} = 85,069. \]
Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for $90 instead.
- Sell the call for $90 and invest $85,069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85,069 = 46,1806$ dollars.
- The fund that remains,

\[ 90 - 85,069 = 4,931 \text{ dollars}, \]

is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 2:
- Suppose the stock price plunges to $120.
- The new delta is 0.25.
- Sell 0.90625 - 0.25 = 0.65625 shares.
- This generates an income of 0.65625 \times 120 = 78.75 dollars.
- Use this income to reduce the debt to $76,04232 \times 1.2 = 78.75 = 12.5$ dollars.
Numerical Examples (continued)

Time 3 (the case of rising price):
- The stock price moves to $180.
- The call we wrote finishes in the money.
- For a loss of $180 - $150 = $30, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + $30 = $45, dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = $45, dollars.

Binomial Tree Algorithms for European Options
- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n^2)$.
  - Can be further reduced to $O(n)$ by reusing space.
- To price European puts, simply replace the payoff,

Numerical Examples (concluded)

Time 3 (the case of declining price):
- The stock price moves to $60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of $0.25 \times 60 = $15, dollars.
- Use it to repay the debt of $12.5 \times 1.2 = $15, dollars.
Further Improvement for Calls

Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that
  \[ b(j; n, p) = \frac{p(n - j + 1)}{(1 - p) j} b(j - 1; n, p). \]
- The following program computes $b(j; n, p)$ in $b[j]$,
  1: $b[a] := \binom{n}{a} p^a (1 - p)^{n - a}$;
  2: for $j = a + 1, a + 2, \ldots, n$ do
  3: \hspace{1em} $b[j] := b[j - 1] \times p \times (n - j + 1)/((1 - p) \times j)$;
  4: end for
- It runs in $O(n)$ steps.

Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (24) on p. 231 is trivial to compute.
- We only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique cannot be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

On the Bushy Tree
Toward the Black-Scholes Formula

- The binomial model suffers from two unrealistic assumptions,
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As the number of periods increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- A proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Use
  \[ \hat{\mu} \equiv \frac{1}{n} E \left[ \ln \frac{S_T}{S} \right] \quad \text{and} \quad \hat{\sigma}^2 \equiv \frac{1}{n} \text{Var} \left[ \ln \frac{S_T}{S} \right] \]
  to denote, resp., the expected value and variance of the period continuously compounded rate of return.
- Under the BOPM, it is not hard to show that
  \[ \hat{\mu} = q \ln(u/d) + \ln d, \]
  \[ \hat{\sigma}^2 = q(1-q) \ln^2(u/d). \]

Toward the Black-Scholes Formula (continued)

- Let \( \tau \) denote the time to expiration of the option measured in years.
- Let \( r \) be the continuously compounded annual rate.
- With \( n \) periods during the option’s life, each period represents a time interval of \( \tau/n \).
- Need to adjust the period-based \( u, d, \) and interest rate \( \hat{r} \) to match the empirical results as \( n \) goes to infinity.
- First, \( \hat{r} = r \tau/n \).
  - The period gross return \( R = e^{\hat{r}} \).

Toward the Black-Scholes Formula (continued)

- Assume the stock’s true continuously compounded rate of return over \( \tau \) years has mean \( \mu_\tau \) and variance \( \sigma^2_\tau \).
  - Call \( \sigma \) the stock’s (annualized) volatility.
- The BOPM converges to the distribution only if
  \[ n\hat{\mu} = n(q \ln(u/d) + \ln d) \to \mu_\tau, \]
  \[ n\hat{\sigma}^2 = nq(1-q) \ln^2(u/d) \to \sigma^2_\tau. \]
- Impose \( ud = 1 \) to make nodes at the same horizontal level of the tree have identical price (review p. 241).
  - Other choices are possible (see text).
Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

\[ u = e^{\sigma \sqrt{\tau/n}}, \quad d = e^{-\sigma \sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\frac{\tau}{n}}. \tag{25} \]

- With Eqs. (25),

\[ \eta \mu = \mu \tau, \]
\[ n \sigma^2 = \left( 1 - \left( \frac{\mu}{\sigma} \right)^2 \frac{\tau}{n} \right) \sigma^2 \tau \rightarrow \sigma^2 \tau. \]

- Other choices are possible (see text).

---

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return \( \ln(S_T/S) \)?

- The central limit theorem says \( \ln(S_T/S) \) converge to the normal distribution with mean \( \mu \tau \) and variance \( \sigma^2 \tau \).

- So \( \ln S_T \) approaches the normal distribution with mean \( \mu \tau + \ln S \) and variance \( \sigma^2 \tau \).

- \( S_T \) has a lognormal distribution in the limit.

---

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities \( u > R > d \) may not hold under Eqs. (25).

- If this happens, the risk-neutral probability may lie outside \([0, 1]\).

- The problem disappears when \( n \) satisfies

\[ e^{\sigma \sqrt{\tau/n}} > e^{\tau/n}, \]

in other words, when \( n > r^2 \tau / \sigma^2 \).

- So it goes away if \( n \) is large enough.

- Other solutions will be presented later.

---

Toward the Black-Scholes Formula (continued)

**Lemma 11** The continuously compounded rate of return \( \ln(S_T/S) \) approaches the normal distribution with mean \( \left( r - \sigma^2/2 \right) \tau \) and variance \( \sigma^2 \tau \) in a risk-neutral economy.

- Let \( q \) equal the risk-neutral probability

\[ p \equiv \frac{e^{\tau/n} - d}{u - d}. \]

- Let \( n \rightarrow \infty \).
Toward the Black-Scholes Formula (continued)

- Lemma 11 and Eq. (18) on p. 151 imply the expected stock price at expiration in a risk-neutral economy is $S e^{r\tau}$.
- The stock's expected annual rate of return is thus the riskless rate $r$.

\*\*In the sense of $(1/\tau) \ln E[S_\tau/S]$ not $(1/\tau) E[\ln(S_\tau/S)]$.

---

BOPM and Black-Scholes Model

- The Black-Scholes formula needs five parameters: $S$, $X$, $\sigma$, $\tau$, and $r$.
- Binomial tree algorithms take six inputs: $S$, $X$, $u$, $d$, $\tilde{r}$, and $n$.
- The connections are
  
  \[
  u = e^{\sigma \sqrt{\tau/n}}, \quad d = e^{-\sigma \sqrt{\tau/n}}, \quad \tilde{r} = r\tau/n,\n  \]

- The binomial tree algorithms converge reasonably fast.
- Oscillations can be eliminated by the judicious choices of $u$ and $d$ (see text).

---

Toward the Black-Scholes Formula (concluded)

**Theorem 12 (The Black-Scholes Formula)**

\[ C = SN(x) - X e^{\tilde{r}\tau} N(x - \sigma \sqrt{\tau}), \]
\[ P = X e^{\tilde{r}\tau} N(-x + \sigma \sqrt{\tau}) - SN(-x), \]

where

\[ x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}. \]
Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market’s opinion of the volatility.
  - Solve for $\sigma$ given the option price, $S$, $X$, $r$, and $r$ with numerical methods,
  - How about American options?
- This volatility is called the implied volatility.
- Implied volatility is often preferred to historical volatility in practice.

Trading Days and Calendar Days (concluded)

- Suppose a year has 260 trading days,
- A quick and dirty way is to replace $\sigma$ with $\sqrt{\frac{365}{260}}$ number of trading days to expiration $\frac{\text{number of calendar days to expiration}}{260}$.
- How about binomial tree algorithms?

\[ \sqrt{\frac{365}{260}} \text{ number of trading days to expiration} \]

\[ \frac{\text{number of calendar days to expiration}}{260} \]

\[ \text{a French (1984).} \]

Trading Days and Calendar Days

- Interest accrues based on the calendar day.
- But $\sigma$ is usually calculated based on trading days only.
  - Stock price seems to have lower volatilities when the exchange is closed.$^a$
- How to incorporate these two different ways of day count into the Black-Scholes formula and binomial tree algorithms?

\[ \text{a Fama (1965); French (1980); French and Roll (1980).} \]

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
  \[ \max(0, X - S\delta^d) \]
  and applies backward induction,
- At each intermediate node, it checks for early exercise by comparing the payoff if exercised with continuation,
Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use $D$ to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$, $(Sd - D)d$.
  - The binomial tree no longer combines.

An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends (Roll, 1977).
- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - $\sigma$ equal to the volatility of the process followed by the risky component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio,
- The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  - A stock that grows from $S$ to $S_T$ with a continuous dividend yield of $q$ would grow from $S$ to $S_T e^{qT}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{qT}$ that pays no dividends.

An Uncompromising Approach\textsuperscript{a}

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases.

\textsuperscript{a}Dai and Lyuu (2004).

Continuous Dividend Yields (continued)

- The Black-Scholes formulas hold with $S$ replaced by $Se^{qT}$ (Merton, 1973):
  \begin{align}
  C &= Se^{qT} N(x) - X e^{rT} N(x - \sigma \sqrt{T}), \\
  P &= X e^{rT} N(-x + \sigma \sqrt{T}) - Se^{qT} N(-x),
  \end{align}
  \quad (26)
  \quad (26')

  where
  \begin{align}
  x &= \frac{\ln(S/X) + (r-q + \sigma^2/2)T}{\sigma \sqrt{T}},
  \end{align}
  \quad (26')

- Formulas (26) and (26') remain valid as long as the dividend yield is predictable.
- Replace $q$ with the average annualized dividend yield.
Continuous Dividend Yields (concluded)

- To run binomial tree algorithms, pick the risk-neutral probability as
  \[ e^{(r - q) \Delta t} - d \]
  \[ u - d \]
  \[ (27) \]
  where \( \Delta t \equiv \tau/n \).
  - Because the stock price grows at an expected rate of \( r - q \) in a risk-neutral economy.
- The \( u \) and \( d \) remain unchanged.
- Other than the change in Eq. (27), binomial tree algorithms stay the same.

Diagonal Traversal of the Tree (Curran, 1995)

- Suppose we traverse the tree diagonally.
- Convince yourself that this procedure is well-defined.
  - An early-exercise node is trivial to evaluate.
  - The difference of the strike price and the stock price.
- A non-early-exercise node must be evaluated by backward induction.

Traversal Sequence

- Can the standard quadratic-time binomial tree algorithm for American options be improved?
  - By an order?
  - By a constant factor?
- In any case, it helps to skip nodes.
- Note the traversal sequence of backward induction on the tree.
  - It is by time (recall p. 240).