Bond Price Volatility

Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest-rate-sensitive securities.
- Assume level-coupon bonds throughout.

Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

\[ \text{Price Volatility} = \frac{\partial P}{\partial y} . \]

Price Volatility of Bonds

- The price volatility of a coupon bond is

\[ \text{Price Volatility} = - \frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)} , \]

where \( F \) is the par value, and \( C \) is the coupon payment per period.

- For bonds without embedded options,

\[ - \frac{\partial P}{\partial y} > 0 . \]
Behavior of Price Volatility (1)

- Price volatility increases as the coupon rate decreases.
  - Zero-coupon bonds are the most volatile.
  - Bonds selling at a deep discount are more volatile than those selling near or above par.
- Price volatility increases as the required yield decreases.
  - So bonds traded with higher yields are less volatile.

Behavior of Price Volatility (2)

- For bonds selling above par or at par, price volatility increases as the term to maturity lengthens (see figure on next page).
  - Bonds with a longer maturity are more volatile.
- For bonds selling below par, price volatility first increases then decreases (see the figure on p. 74).
  - Longer maturity here cannot be equated with higher price volatility.
Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' PVs divided by the asset's price.
- Formally,
  \[ MD = \frac{1}{F} \sum_{i=1}^{n} \frac{iC_i}{(1+y)^i}. \]
- The Macaulay duration, in periods, is equal to
  \[ MD = -(1+y) \frac{\partial P}{\partial y} F. \]  
(7)

Finesse

- Equations (7) on p. 75 and (8) on p. 76 hold only if the coupon \( C \), the par value \( F \), and the maturity \( n \) are all independent of the yield \( y \).
- That is, if the cash flow is independent of yields.

MD of Bonds

- The MD of a coupon bond is
  \[ MD = \frac{1}{F} \left[ \sum_{i=1}^{n} \frac{iC_i}{(1+y)^i} + \frac{nF}{(1+y)^n} \right]. \]  
(8)
- It can be simplified to
  \[ MD = \frac{c(1+y)[(1+y)^n - 1] + ny(y-c)}{cy[(1+y)^n - 1] + y^2}, \]
  where \( c \) is the period coupon rate.
- The MD of a zero-coupon bond equals its term to maturity \( n \).
- The MD of a coupon bond is less than its maturity.

How Not To Think of MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But you use it that way at your peril.
- The MD should be seen mainly as measuring price volatility.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.
Modified Duration

- Modified duration is defined as
  \[
  \text{modified duration} = -\frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1+y)}. \tag{9}
  \]
- By Taylor expansion,
  percent price change \( \approx -\text{modified duration} \times \text{yield change}. \)

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be
  \[ -11.54 \times 0.001 = -0.01154 = -1.154\%. \]

Modified Duration of a Portfolio

- The modified duration of a portfolio equals
  \[
  \sum_i \omega_i D_i.
  \]
- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.

Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as
  \[
  \frac{P_+ - P_-}{P_0(y_+ - y_-)},
  \]
- \( P \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.
**Effective Duration (concluded)**

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use
  \[ \frac{P_0 - P_+}{P_0 \Delta y}. \]
  - More economical but less accurate.

**Meeting Liabilities**

- Buy coupon bonds to meet a future liability.
- What happens at the horizon date when the liability is due?
- Say interest rates rise subsequent to the purchase:
  - The interest on interest from the reinvestment of the coupon payments will increase,
  - But a capital loss will occur for the sale of the bonds.
- The reverse is true if interest rates fall.
- Uncertainties in meeting the liability.

**The Practices**

- Duration is usually expressed in percentage terms—call it \( D\% \) for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by
  \[ -D\% \times \Delta r \]
  when the yield increases instantaneously by \( \Delta r\% \).
  - Price will drop by 20% if \( D\% = 10 \) and \( \Delta r = 2 \) because \( 10 \times 2 = 20 \).
- In fact, \( D\% \) equals modified duration as originally defined [prove it!].

**Immunization**

- A portfolio immunizes a liability if its value at horizon covers the liability for small rate changes now.
- A bond portfolio whose MD equals the horizon and whose PV equals the PV of the single future liability,
  - At horizon, losses from the interest on interest will be compensated by gains in the sale price when interest rates fall,
  - Losses from the sale price will be compensated by the gains in the interest on interest when interest rates rise [see figure on p. 87].
The Proof

- Assume the liability is \( L \) at time \( m \) and the current interest rate is \( y \).
- Want a portfolio such that
  1. Its FV is \( L \) at the horizon \( m \);
  2. \( \frac{\partial \text{FV}}{\partial y} = 0 \);
  3. FV is convex around \( y \).
- Condition (1) says the obligation is met.
- Conditions (2) and (3) mean \( L \) is the portfolio’s minimum FV at horizon for small rate changes.

The Proof (continued)

- Let \( \text{FV} \equiv (1 + y)^m \text{P} \), where \( \text{P} \) is the PV of the portfolio.
- Now,
  \[
  \frac{\partial \text{FV}}{\partial y} = m(1 + y)^{m-1} \text{P} + (1 + y)^m \frac{\partial \text{P}}{\partial y}.
  \]
- Imposing Condition (2) leads to
  \[
  m = -(1 + y) \frac{\partial \text{P}/\text{P}}{\partial y}.
  \]
- The MD is equal to the horizon \( m \).
The Proof (concluded)

- Employ a coupon bond for immunization.

- Since

\[ FV = \sum_{i=1}^{n} \frac{C}{(1 + y)^{i-m}} + \frac{F}{(1 + y)^n}, \]

it follows that

\[ \frac{\partial^2 FV}{\partial y^2} > 0 \]  \hspace{1cm} (10)

for \( y > -1 \).

- Since FV is convex for \( y > -1 \), the minimum value of FV is indeed \( L \).

Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

\[ \text{modified duration} \times \text{price (\% of par)} = -\frac{\partial P}{\partial y}. \]

- The approximate dollar price change per $100 of par value is

\[ \text{price change} \approx -\text{dollar duration} \times \text{yield change}. \]

Rebalancing

- Immunization has to be rebalanced constantly to ensure that the MD remains matched to the horizon.

- The MD decreases as time passes.

- But, except for zero-coupon bonds, the decrement is not identical to that in the time to maturity.
  - Consider a coupon bond whose MD matches horizon.
  - Since the bond's maturity date lies beyond the horizon date, its MD will remain positive at horizon.

- So immunization needs to be reestablished even if interest rates never change.

Convexity

- Convexity is defined as

\[ \text{convexity (in periods)} = \frac{\partial^2 P}{\partial y^2} \frac{1}{P}. \]

- The convexity of a coupon bond is positive (see Eq. (10) on p. 91).

- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude.

- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.
Use of Convexity

- The approximation $\Delta P/P \approx -\text{duration} \times \text{yield change}$ works for small yield changes.
- To improve upon it for larger yield changes, use
  \[
  \frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2
  \]
  \[
  = -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.
  \]
- Recall the figure on p. 95.

Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by
  \[
  \text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
  \]
  when there are $k$ periods per annum.
- The convexity of a coupon bond increases as its coupon rate decreases.
- For a given yield and duration, the convexity decreases as the coupon decreases.

The Practices

- Convexity is usually expressed in percentage terms—call it $C_p$—for quick mental calculation.
- The percentage price change expressed in percentage terms is approximated by $-D_p \times \Delta r + C_p \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
- Price will drop by 17% if $D_p = 10$, $C_p = 1.5$, and $\Delta r = 2$ because
  \[
  -10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.
  \]
- In fact, $C_p$ equals convexity divided by 100 (prove it!).
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds forms the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.
- The term structure is fundamental to the valuation of fixed-income securities.

Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.
Four Shapes
- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Problems with the PV Formula
- In the bond price formula,
  \[
  P = \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},
  \]
  every cash flow is discounted at the same yield \( y \).
- Consider two riskless bonds with different yields to maturity because of their different cash flow streams,
- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates,
- But shouldn't they be discounted at the same rate?
- Enter the spot rate methodology.

Spot Rates
- The \( i \)-period spot rate \( S(i) \) is the yield to maturity of an \( i \)-period zero-coupon bond.
- The PV of one dollar \( i \) periods from now is
  \[ [1 + S(i)]^{-i}. \]
- The one-period spot rate is called the short rate.
- A spot rate curve is a plot of spot rates against maturity.

Spot Rate Discount Methodology
- A cash flow \( C_1, C_2, \ldots, C_n \) is equivalent to a package of zero-coupon bonds with the \( i \)th bond paying \( C_i \) dollars at time \( i \).
- So a level-coupon bond has the price
  \[
  P = \sum_{i=1}^{n} \frac{C_i}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n},
  \] (11)
- This pricing method incorporates information from the term structure.
- Discount each cash flow at the corresponding spot rate.
Discount Factors

- In general, any riskless security having a cash flow \( C_1, C_2, \ldots, C_n \) should have a market price of

\[
P = \sum_{i=1}^{n} C_i d(i).
\]

- Above, \( d(i) \equiv \left[1 + S(i)\right]^{-i} \), \( i = 1, 2, \ldots, n \), are called discount factors.
- \( d(i) \) is the PV of one dollar \( i \) periods from now.
- The discount factors are often interpolated to form a continuous function called the discount function.

Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price \( P \) of the \( n \)-period coupon bond and \( S(1), S(2), \ldots, S(n-1) \).
- Then \( S(n) \) can be computed from Eq. (11), repeated below,

\[
P = \sum_{i=1}^{n} \frac{C}{\left[1 + S(i)\right]^i} + \frac{F}{\left[1 + S(n)\right]^n}.
\]

- The running time is \( O(n) \).
- The procedure is called bootstrapping.

Extracting Spot Rates from Yield Curve

- Start with the short rate \( S(1) \).
- Note that short-term Treasuries are zero-coupon bonds.
- Compute \( S(2) \) from the two period coupon bond price \( P \) by solving

\[
P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.
\]

Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
- Lack economic justifications.
Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \ldots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \ldots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \ldots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \ldots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Shapes

- The spot rate curve often has the same shape as the yield curve.
- If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But only a trend not a mathematical truth.

Coupon Effect on the Yield to Maturity

- Under a normal spot rate curve, a coupon bond has a lower yield than a zero-coupon bond of equal maturity.
- Picking a zero-coupon bond over a coupon bond based purely on the zero’s higher yield to maturity is flawed.

Shapes (concluded)

- When the final principal payment is relatively insignificant, the spot rate curve and the yield curve do share the same shape.
  - Bonds of high coupon rates and long maturities.
- By the agreement in shape, remember the above proviso.
Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.
- Invest $1 for $j$ periods to end up with $[1 + S(j)]^j$ dollars at time $j$.
  - The maturity strategy.
- Invest $1 in bonds for $i$ periods and at time $i$ invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[1 + S(i)]^i[1 + S(i,j)]^j$ dollars at time $j$.
  - $S(i,j)$: $(j - i)$-period spot rate $i$ periods from now.
  - The rollover strategy.

Forward Rates (concluded)

- When $S(i,j)$ equals
  \[
  f(i,j) = \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j - i)} - 1, \tag{12}
  \]
  we will end up with $[1 + S(j)]^j$ dollars again.
- By definition, $f(0, j) = S(j)$.
- $f(i, j)$ is called the (implied) forward rates.
  - More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,
  \[ f(i, j) > S(j) > \cdots > S(i). \]
- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,
  \[ f(i, j) < S(j) < \cdots < S(i). \]

Forward Rates = Spot Rates = Yield Curve

- The FV of $1 at time \( n \) can be derived in two ways,
- Buy \( n \)-period zero-coupon bonds and receive \( [1 + S(n)]^n \).
- Buy one period zero coupon bonds today and a series of such bonds at the forward rates as they mature,
- The FV is \( [1 + S(1)](1 + f(1, 2)) \cdots [1 + f(n - 1, n)] \).

Forward Rates = Spot Rates = Yield Curve (concluded)

- Since they are identical,
  \[
  S(n) = (1 + S(1))(1 + f(1, 2))
  \cdots (1 + f(n - 1, n)))^{1/n} - 1, \quad (13)
  \]
- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.
- Other equivalency can be derived similarly.
- Show that \( f(T, T + 1) = d(T)/d(T + 1) = 1 \).
Locking in the Forward Rate $f(n, m)$

- Buy one $n$-period zero-coupon bond for $1/(1 + S(n))^n$.
- Sell $(1 + S(m))^m/(1 + S(n))^n$ $m$-year zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1 + S(n))^n$.
- At time $n$ there will be a cash inflow of $1$.
- At time $m$ there will be a cash outflow of $(1 + S(m))^m/(1 + S(n))^n$ dollars.
- This implies the rate $f(n,m)$ between times $n$ and $m$.

Forward Contracts

- We generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to borrow money at time $n$ in the future and repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.
- Can the spot rate curve be an arbitrary curve?

Spot and Forward Rates under Continuous Compounding

- The pricing formula:
  $$P = \sum_{i=1}^{n} Ce^{-iS[i]} + Fe^{-nS[n]}.$$  
- The market discount function:
  $$d(n) = e^{-nS(n)}.$$  
- The spot rate is an arithmetic average of forward rates,
  $$S(n) = \frac{f(0,1) + f(1,2) + \ldots + f(n-1,n)}{n}.$$
Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:
  \[ f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \]
- The one-period forward rate:
  \[ f(j, j + 1) = -\ln \frac{d(j + 1)}{d(j)}. \]
- \[ f(T) \equiv \lim_{\Delta T \to 0} f(T, T + \Delta T) = S(T) + T \frac{\partial S}{\partial T}. \]
- \[ f(T) > S(T) \text{ if and only if } \frac{\partial S}{\partial T} > 0. \]

Unbiased Expectations Theory and Spot Rate Curve

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,
  \[ f(a, b) = E[S(a, b)]. \]
- Does not imply that the forward rate is an accurate predictor for the future spot rate.
- Implies that the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.
Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:
  \[
  E \left[ \frac{(1 + S(1,n))^{(n-1)}}{(1 + S(n))^{n}} \right] = 1 + S(1) \quad \text{for all } n > 1.
  \]
- This theory is the basis of many interest rate models.
- Holding premium:
  \[
  E \left[ \frac{(1 + S(1,n))^{(n-1)}}{(1 + S(n))^{n}} \right] - (1 + S(1)).
  \]
  - Zero under the local expectations theory.

Duration Revisited

- Let \( P(y) \equiv \sum_i C_i / (1 + S(i) + y)^i \) be the price associated with the cash flow \( C_1, C_2, \ldots \).
- Define duration as
  \[
  - \left. \frac{\partial P(y)}{\partial y} \right|_{y=0} = \sum_i \frac{C_i}{(1 + S(i) + y)^i} \times \frac{1}{(1 + S(i))^i}.
  \]
  - The curve is shifted in parallel to \( S(1) + \Delta y, S(2) + \Delta y, \ldots \) before letting \( \Delta y \) go to zero.
- The percentage price change roughly equals duration times the size of the parallel shift in the spot rate curve.

Duration Revisited (continued)

- The simple linear relation between duration and MD in Eq. (9) on p. 79 breaks down.
- One way to regain it is to resort to a different kind of shift, the proportional shift:
  \[
  \frac{\Delta (1 + S(i))}{1 + S(i)} = \frac{\Delta (1 + S(1))}{1 + S(1)}
  \]
  for all \( i \).
  - \( \Delta (x) \) denotes the change in \( x \) when the short-term rate is shifted by \( \Delta y \).

Duration Revisited (concluded)

- Duration now becomes
  \[
  \frac{1}{1 + S(1)} \left[ \sum_i \frac{i C_i (1 + S(i))^i}{(1 + S(1))^i} \right].
  \]
  (15)
- Define Macaulay’s second duration to be the number within the brackets in Eq. (15).
- Then
  \[
  \text{duration} = \frac{\text{Macaulay’s second duration}}{(1 + S(1))}.
  \]
**Immunization Revisited**

- Recall that a future liability can be immunized by matching PV and MD under flat spot rate curves.
- If only parallel shifts are allowed, this conclusion continues to hold under general spot rate curves.
- Assume liability $L$ is $T$ periods from now.
- Assume $L = 1$ for simplicity.
- Assume the matching portfolio consists only of zero-coupon bonds maturing at $t_1$ and $t_2$ with $t_1 < T < t_2$.

**Immunization Revisited (concluded)**

- Now shift the spot rate curve uniformly by $\delta \neq 0$.
- The portfolio's PV becomes
  \[ n_1 e^{(S(t_1) + \delta) t_1} + n_2 e^{(S(t_2) + \delta) t_2} = e^{\delta t_1} V(t_2 - T) + e^{\delta t_2} V(t_1 - T) \]
  \[ = e^{\delta t_1} \frac{V(t_2 - T)}{t_2 - t_1} + e^{\delta t_2} \frac{V(t_1 - T)}{t_1 - t_2} \]
  \[ = \frac{V}{t_2 - t_1} (e^{\delta t_1} (t_2 - T) + e^{\delta t_2} (T - t_1)) \).
- The liability's PV after shift is $e^{-(S(T) + \delta) T} = e^{-\delta TV}$.
- And $\frac{V}{t_2 - t_1} (e^{\delta t_1} (t_2 - T) + e^{\delta t_2} (T - t_1)) > e^{-\delta TV}$.

**Immunization Revisited (continued)**

- Let there be $n_i$ bonds maturing at time $t_i$, $i = 1, 2$.
- The portfolio's PV is
  \[ V \equiv n_1 e^{-S(t_1) t_1} + n_2 e^{-S(t_2) t_2} = e^{-S(T) T} \]
- Its MD is
  \[ \frac{n_1 t_1 e^{S(t_1) t_1} + n_2 t_2 e^{S(t_2) t_2}}{V} = T, \]
- These two equations imply
  \[ n_1 e^{S(t_1) t_1} = \frac{V(t_2 - T)}{t_2 - t_1} \quad \text{and} \quad n_2 e^{S(t_2) t_2} = \frac{V(t_1 - T)}{t_1 - t_2}. \]
Two Intriguing Implications

- A duration-matched position under parallel shifts implies free lunch as any interest rate change generates profits.
- No investors would hold the $T$ period bond because a portfolio of $t_1$- and $t_2$-period bonds has a higher return for any interest rate shock,
  - They would own only bonds of the shortest and longest maturities.
- The logic seems impeccable,
- What gives?