Efficient Algorithms for PV and FV

- The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is
  \[ \frac{C_1}{1 + y} + \frac{C_2}{(1 + y)^2} + \cdots + \frac{C_n}{(1 + y)^n}. \]
- This formula and its variations are the engine behind most of financial calculations.
  - What is $y$?
  - What are $C_i$?
  - What is $n$?
- It can be computed by the algorithm on p. 28.

The Idea Behind p. 28: Homer’s Rule

- This idea is
  \[ \frac{1}{1 + y} \left( \frac{C_n}{1 + y^{n-1}} \right) \left( \frac{1}{1 + y^{n-2}} \right) \left( \frac{1}{1 + y} \right). \]
  - Due to Homer (1786-1837) in 1819.
- The algorithm takes $O(n)$ time.
- It is the most efficient possible in terms of the absolute number of arithmetic operations.

Algorithm for Evaluating PV

1: $x := 0$
2: $d := 1 + y$
3: for $i = n, n-1, \ldots, 1$ do
4: \hspace{1em} $x := (x + C_i)/d$
5: end for
6: return $x$

Conversion between Compounding Methods

- Suppose $r_1$ is the annual rate with continuous compounding.
- Suppose $r_2$ is the equivalent rate compounded $m$ times per annum.
- How are they related?
Conversion between Compounding Methods (concluded)

- Both interest rates must produce the same amount of money after one year.
- That is,
  \[
  (1 + \frac{r_2}{m})^m = e^{r_1}.
  \]
- Therefore,
  \[
  r_1 = m \ln \left(1 + \frac{r_2}{m}\right),
  \]
  \[
  r_2 = m \left(e^{r_1/m} - 1\right).
  \]

Annuities

- An annuity pays out the same \( C \) dollars at the end of each year for \( n \) years.
- With a rate of \( r \), the FV at the end of the \( n \)th year is
  \[
  \sum_{i=0}^{n-1} (1 + r)^i = C \frac{(1 + r)^n - 1}{r}.
  \]  

But Are They Really “Equivalent”? 

- Recall \( r_1 \) and \( r_2 \) on the previous page.
- They are based on different cash flows.
- In what sense are they equivalent?

General Annuities

- If \( m \) payments of \( C \) dollars each are received per year (the general annuity), then Eq. (2) becomes
  \[
  C \frac{(1 + \frac{r}{m})^{nm} - 1}{\frac{r}{m}}.
  \]
- The PV of a general annuity is
  \[
  \sum_{i=1}^{nm} C \left(1 + \frac{r}{m}\right)^i = C \frac{1 - (1 + \frac{r}{m})^{nm}}{\frac{r}{m}}.
  \]
Amortization

- It is a method of repaying a loan through regular payments of interest and principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

A Numerical Example

- Consider a 15-year, $250,000 loan at 8.0% interest rate.
- Solving Eq. (3) on p. 34 with \( PV = 250000, n = 15, m = 12, \) and \( r = 0.08 \) gives a monthly payment of \( C = 2389.13. \)
- The amortization schedule is shown on p. 38.
- In every month (1) the principal and interest parts add up to $2,389.13, (2) the remaining principal is reduced by the amount indicated under the Principal heading, and (3) the interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages.
  - They are called traditional mortgages in the U.S.

<table>
<thead>
<tr>
<th>Month</th>
<th>Payment</th>
<th>Interest</th>
<th>Principal</th>
<th>Remaining principal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2,389.13</td>
<td>1,666.67</td>
<td>722,464</td>
<td>249,277,536</td>
</tr>
<tr>
<td>2</td>
<td>2,389.13</td>
<td>1,661,850</td>
<td>727,280</td>
<td>248,550,256</td>
</tr>
<tr>
<td>3</td>
<td>2,389.13</td>
<td>1,657,002</td>
<td>732,129</td>
<td>247,818,128</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td>47,153</td>
<td>2,341,980</td>
<td>4,730,899</td>
</tr>
<tr>
<td>178</td>
<td>2,389.13</td>
<td>31,539</td>
<td>2,357,591</td>
<td>2,373,308</td>
</tr>
<tr>
<td>179</td>
<td>2,389.13</td>
<td>15,822</td>
<td>2,373,308</td>
<td>0,000</td>
</tr>
<tr>
<td>180</td>
<td>2,389.13</td>
<td>15,822</td>
<td>2,373,308</td>
<td>0,000</td>
</tr>
<tr>
<td>Total</td>
<td>430,043,438</td>
<td>180,043,438</td>
<td>250,000,000</td>
<td></td>
</tr>
</tbody>
</table>
Two Methods of Calculating the Remaining Principal

1. Go down the amortization schedule.
   - This method is relatively slow, however.
2. Right after the kth payment, the remaining principal is the PV of the future \( nm - k \) cash flows,
   \[
   \sum_{i=1}^{nm-k} C \left( 1 + \frac{r}{m} \right)^i = C \left( 1 + \frac{r}{m} \right)^{nm-k}.
   \]

Internal Rate of Return (IRR)

- It is the interest rate which equates an investment’s PV with its price \( P \),
  \[
  P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_n}{(1+y)^n}.
  \]
- IRR assumes all cash flows are reinvested at the same rate as the internal rate of return.
- The above formula is the foundation upon which pricing methodologies are built.

Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- BEY corresponds to the \( r \) in Eq. (1) on p. 22 that equates PV with FV when \( m = 2 \).
- MEY corresponds to the \( r \) in Eq. (1) on p. 22 that equates PV with FV when \( m = 12 \).

Holding Period Return (HPR)

- Calculate the FV by whatever means and then find the yield \( y \) that satisfies \( PV = FV \times e^{-y^n} \).
- Explicit assumptions about the reinvestment rates must be made.
- If the reinvestment assumptions turn out to be wrong, the yield will not be realized.
  - This is the reinvestment risk.
- Financial instruments without intermediate cash flows do not have reinvestment risks.
Numerical Methods for Yields

- Solve \( f(y) = 0 \) for \( y \geq -1 \), where
  \[
  f(y) = \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P.
  \]
  - \( P \) is the market price,
- The function \( f(y) \) is monotonic in \( y \) if \( C_t > 0 \) for all \( t \).
- A unique solution exists for a monotonic \( f(y) \).

The Newton-Raphson Method

- Converges faster than the bisection method.
- Start with a first approximation \( x_0 \) to a root of \( f(x) = 0 \).
- Then
  \[
  x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.
  \]
- When computing yields,
  \[
  f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^t+1}.
  \]

The Bisection Method

- Start with \( a \) and \( b \) where \( a < b \) and \( f(a) f(b) < 0 \).
- Then \( f(\xi) \) must be zero for some \( \xi \in [a, b] \).
- If we evaluate \( f \) at the midpoint \( c \equiv (a + b)/2 \), either
  1. \( f(c) = 0 \),
  2. \( f(a) f(c) < 0 \), or
  3. \( f(c) f(b) < 0 \).
- In the first case we are done, in the second case we continue the process with the new bracket \([a, c]\), and in the third case we continue with \([c, b]\).
- The bracket is halved in the latter two cases.
- After \( n \) steps, we will have confined \( \xi \) within a bracket of length \((b-a)/2^n\).
The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations \(x_0\) and \(x_1\).
- Then compute the \((k+1)\)st approximation with
  \[x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \]
- Its convergence rate, 1.618, is slightly worse than the Newton-Raphson method’s 2 but better than the bisection method’s 1.

Solving Systems of Nonlinear Equations (concluded)

- The \((k+1)\)st approximation \((x_{k+1}, y_{k+1})\) satisfies the following linear equations,
  \[
  \begin{bmatrix}
  \frac{\partial f(x_k, y_k)}{\partial x} & \frac{\partial f(x_k, y_k)}{\partial y} \\
  \frac{\partial g(x_k, y_k)}{\partial x} & \frac{\partial g(x_k, y_k)}{\partial y}
  \end{bmatrix}
  \begin{bmatrix}
  \Delta x_{k+1} \\
  \Delta y_{k+1}
  \end{bmatrix}
  =
  \begin{bmatrix}
  f(x_k, y_k) \\
  g(x_k, y_k)
  \end{bmatrix},
  \]
  where \(\Delta x_{k+1} \equiv x_{k+1} - x_k\) and \(\Delta y_{k+1} \equiv y_{k+1} - y_k\).
- The above has a unique solution for \((\Delta x_{k+1}, \Delta y_{k+1})\) when the \(2 \times 2\) matrix is invertible.
- Set \((x_{k+1}, y_{k+1}) = (x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})\).

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- But the Newton-Raphson method can be extended to higher dimensions.
- Let \((x_k, y_k)\) be the \(k\)th approximation to the solution of the two simultaneous equations,
  \[
  f(x, y) = 0, \\
  g(x, y) = 0,
  \]

Zero-Coupon Bonds (Pure Discount Bonds)

- The price of a zero-coupon bond that pays \(F\) dollars in \(n\) periods is
  \[F/(1 + r)^n,\]
  where \(r\) is the interest rate per period.
- Can meet future obligations without reinvestment risk.
- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.
Example
- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at \(1/(1.04)^{40}\), or 20.83%, of its par value.
- It will be quoted as 20.83.
- If the bond matures in 10 years instead of 20, its price would be 45.64.

Pricing Formula
\[
P = \sum_{i=1}^{n} \frac{C}{(1 + \frac{r}{m})^i} + \frac{F}{(1 + \frac{r}{m})^n}
\]
\[
= C \frac{1}{(1 + \frac{r}{m})^n} + \frac{F}{(1 + \frac{r}{m})^n}
\]

- \(n\): number of cash flows.
- \(m\): number of cash flows per year.
- \(r\): annual rate compounded \(m\) times per annum.
- \(C = Fc/m\) when \(c\) is the annual coupon rate.

Level-Coupon Bonds
- Coupon rate.
- Par value, paid at maturity.
- \(F\) denotes the par value and \(C\) denotes the coupon.
- Cash flow:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(C)</td>
<td>(C)</td>
<td>(C)</td>
<td>(C + F)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>(n)</td>
</tr>
</tbody>
</table>

Yields to Maturity
- The \(r\) that satisfies Eq. (4) on p. 53 with \(P\) being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

\[
5 \times \frac{1 - [1 + (0.15/2)]^{2 \times 10}}{0.15/2} + \frac{100}{[1 + (0.15/2)]^{2 \times 10}} = 74,5138\]

percent of par,
Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- “Only 24 percent answered the question correctly.”

*CNN, December 21, 2001.*

<table>
<thead>
<tr>
<th>Yield (%)</th>
<th>Price (% of par)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5</td>
<td>113.37</td>
</tr>
<tr>
<td>8.0</td>
<td>108.65</td>
</tr>
<tr>
<td>8.5</td>
<td>104.19</td>
</tr>
<tr>
<td>9.0</td>
<td>100.00</td>
</tr>
<tr>
<td>9.5</td>
<td>96.04</td>
</tr>
<tr>
<td>10.0</td>
<td>92.31</td>
</tr>
<tr>
<td>10.5</td>
<td>88.79</td>
</tr>
</tbody>
</table>

Price Behavior (2)

- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

**Terminology**

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.
Day Count Conventions: 30/360
- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date \( D_1 \equiv (y_1, m_1, d_1) \) to date \( D_2 \equiv (y_2, m_2, d_2) \) is
  \[
  360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1).
  \]

Day Count Conventions: Actual/Actual
- The first “actual” refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period,
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Full Price (Dirty Price, Invoice Price)
- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let
  \[
  \omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}.
  \]
- The price is now calculated by
  \[
  PV = \sum_{i=0}^{n} \frac{C}{(1 + \frac{r}{m})^{m+i}} + \frac{F}{(1 + \frac{r}{m})^{m+n}}.
  \]
Accrued Interest

- The buyer pays the quoted price plus the accrued interest

\[ C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 + \omega). \]

- The yield to maturity is the \( r \) satisfying (6) when \( P \) is the invoice price, the sum of the quoted price and the accrued interest.

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.

Example ("30/360") (concluded)

- The accrued interest is \( (10/2) \times \frac{180 - 60}{180} = 3.3333 \) per $100 of par value.

- The yield to maturity is 3%.

- This can be verified by Eq. (6) with \( \omega = 60/180, \ m = 2, \ C = 5, \ PV = 111.2891 + 3.3333, \) and \( r = 0.03. \)

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.

- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.

- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.

Price Behavior (2) Revisited

- Before: A bond selling at par if the yield to maturity equals the coupon rate.

- But it assumed that the settlement date is on a coupon payment date.

- Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.

- Then its yield to maturity is less than the coupon rate,
  - The short reason: Exponential growth is replaced by linear growth, hence “overpaying” the coupon.