Impacts of the Initial Conditions

- Different initial conditions give rise to different solutions.
- Suppose $a_0 = 1$ and $a_1 = 2$.
- Then solve

\[
1 = a_0 = c_1 + c_2,
\]
\[
2 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\]
for

\[
c_1 = \frac{[(1 + \sqrt{5})/2]^2}{\sqrt{5}},
\]
\[
c_2 = -\frac{[(1 - \sqrt{5})/2]^2}{\sqrt{5}}.
\]
Impacts of the Initial Conditions (continued)

- Finally,

\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} \]  \hspace{1cm} (82)
Impacts of the Initial Conditions (continued)

- Suppose $a_0 = a_1 = 1$ instead.

- Then solve

\[
1 = a_0 = c_1 + c_2,
\]

\[
1 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\]

for

\[
c_1 = \left[ \frac{(1 + \sqrt{5})}{2} \right]/\sqrt{5},
\]

\[
c_2 = -\left[ \frac{(1 - \sqrt{5})}{2} \right]/\sqrt{5}.
\]
Impacts of the Initial Conditions (concluded)

• Finally,

\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \] . \quad (83)

• This formula differs from Eq. (82) on p. 555.
Generating Function for the Fibonacci Relation

- From $a_{n+2} = a_{n+1} + a_n$, we obtain
  \[ \sum_{n=0}^{\infty} a_{n+2}x^{n+2} = \sum_{n=0}^{\infty} \left( a_{n+1}x^{n+2} + a_nx^{n+2} \right). \]

- Let $f(x)$ be the generating function for \{ $a_n$ \}_{n=0,1,2,...}.

- Then
  \[ f(x) - a_0 - a_1x = x[f(x) - a_0] + x^2f(x). \]

- Hence
  \[ f(x) = \frac{-a_0x + a_0 + a_1x}{1 - x - x^2}. \quad (84) \]
A Formula for the Fibonacci Numbers $a_n$

\[
\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n = \sum_{m=0}^{\lfloor n/2 \rfloor - 1} \binom{n - m - 1}{m}.
\]

- The generating function (84) on p. 558 gives

\[
\frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2} = \frac{x}{1 - x(1 + x)} = x + x^2(1 + x) + x^3(1 + x)^2 + \cdots
\]

\[
+ x^{n-1}(1 + x)^{n-2} + x^n(1 + x)^{n-1} + \cdots
\]

\[
= \cdots + \left[ \binom{n - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor - 1} + \cdots + \binom{n - 2}{1} + \binom{n - 1}{0} \right] x^n + \cdots.
\]

\[\text{aRecall that } a_0 = 0 \text{ and } a_1 = 1.\]
Binary Sequences without Consecutive 0s

- Let $a_n$ denote the number of binary sequences of length $n$ without consecutive 0s.

- There are $a_{n-1}$ valid sequences with the $n$th symbol being 1.

- There are $a_{n-2}$ valid sequences with the $n$th symbol being 0 because any such sequence must end with 10.

- Hence $a_n = a_{n-1} + a_{n-2}$, a Fibonacci sequence.

- Because $a_1 = 2$ and $a_2 = 3$, we must have $a_0 = 1$ to retrofit the Fibonacci sequence.

- The formula is Eq. (82) on p. 555.
Number of Subsets without Consecutive Numbers

• How many subsets of \( \{1, 2, \ldots, n\} \) contain no 2 consecutive integers?

• A binary sequences \( b_1 b_2 \cdots b_n \) of length \( n \) can be interpreted as the set \( \{i : b_i = 0\} \subseteq \{1, 2, \ldots, n\} \).

• So a subset of \( \{1, 2, \ldots, n\} \) without consecutive integers implies a binary sequence without consecutive 0s, and vice versa.

• Hence there are \( a_n \) subsets of \( \{1, 2, \ldots, n\} \) that contain no 2 consecutive integers, where \( a_n \) is the Fibonacci number with \( a_0 = 1 \) and \( a_1 = 2 \).\(^a\)

\(^a\)Recall p. 560.
Number of Subsets without Consecutive Numbers (continued)

• From formula (82) on p. 555,

\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^{n+2} \]

is the Fibonacci number with \( a_0 = 1 \) and \( a_1 = 2 \).

• We knew there are \( \binom{n-m+1}{m} \) \( m \)-element subsets of \( \{1, 2, \ldots, n\} \) that contain no consecutive integers.\(^a\)

\(^a\)Recall Eq. (16) on p. 95.
Number of Subsets without Consecutive Numbers (concluded)

- Hence $a_n$ also equals

$$\sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m+1}{m}.$$  

- In summary,

$$\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m+1}{m}.$$  

- We could have used the identity on p. 559 to derive it.
Number of Subsets without Cyclically Consecutive Numbers

• How many subsets of \( \{1, 2, \ldots, n\} \) contain no 2 consecutive integers when 1 and \( n \) are considered consecutive?

• Let \( a_n \) be the solution for the problem on p. 561.

• So \( a_n \) is the Fibonacci number with \( a_0 = 1 \) and \( a_1 = 2 \) (formula appeared in Eq. (82) on p. 555).

• Now assume \( n \geq 3 \) first.

• There are \( a_{n-1} \) acceptable subsets that do not contain \( n \).
Number of Subsets without Cyclically Consecutive Numbers (continued)

- If \( n \) is included, an acceptable subset cannot contain 1 or \( n - 1 \).
- Hence there are \( a_{n-3} \) such subsets.
- The total is therefore \( L_n \triangleq a_{n-1} + a_{n-3} \), the **Lucas number**.\(^a\)
- It can be easily checked that

\[
L_n = a_{n-1} + a_{n-3} \\
= a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5} \\
= L_{n-1} + L_{n-2}.
\]

\(^a\)Corrected by Mr. Gong-Ching Lin (B00703082) on May 19, 2012.
Number of Subsets without Cyclically Consecutive Numbers (continued)

- Furthermore, $L_0 = 2$ and $L_1 = 1$.
  - $L_3 = a_2 + a_0 = 3 + 1 = 4$ and $L_4 = a_3 + a_1 = 5 + 2 = 7$.
  - So

\[
\begin{align*}
L_2 &= L_4 - L_3 = 3, \\
L_1 &= L_3 - L_2 = 1, \\
L_0 &= L_2 - L_1 = 2.
\end{align*}
\]
Number of Subsets without Cyclically Consecutive Numbers (continued)

- The general solution is

\[ L_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

by Eq. (80) on p. 547.

- Solve

\[
\begin{align*}
2 &= L_0 = c_1 + c_2, \\
1 &= L_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\end{align*}
\]

for \( c_1 = 1 \) and \( c_2 = 1 \).
Number of Subsets without Cyclically Consecutive Numbers (concluded)

• The solution is finally

\[ L_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n. \]
Number of Palindromes Revisited

- A palindrome is a composition for $n \in \mathbb{Z}^+$ that reads the same left to right as right to left (p. 110).
- Let $a_n$ denote the number of palindromes for $n$.
- Clearly, $a_1 = 1$ and $a_2 = 2$.
- Given each palindrome for $n$, we can do two things to obtain a palindrome for $n + 2$.
  - Add 1 to the first and last summands.
    * So $1 + 3 + 1$ becomes $2 + 3 + 2$.
  - Insert summand 1 to the start and end.
    * So $1 + 3 + 1$ becomes $1 + 1 + 3 + 1 + 1$. 
The Proof (continued)

- This mapping is a one-to-one correspondence (why?).
- Hence
  \[ a_{n+2} = 2a_n, \quad n \geq 1. \]
- The characteristic equation
  \[ r^2 - 2 = 0 \]
  has two roots \( \pm \sqrt{2} \).
The Proof (continued)

• The general solution is hence

\[a_n = c_1 \left(\sqrt{2}\right)^n + c_2 \left(-\sqrt{2}\right)^n.\]

• Solve\(^a\)

\[
\begin{align*}
1 &= a_1 = \sqrt{2}\left(c_1 - c_2\right), \\
2 &= a_2 = 2\left(c_1 + c_2\right),
\end{align*}
\]

for \(c_1 = \left(1 + \frac{1}{\sqrt{2}}\right)/2\) and \(c_2 = \left(1 - \frac{1}{\sqrt{2}}\right)/2.\)

\(^a\)This time, we are not retrofitting.
The Proof (concluded)

- The number of palindromes for $n$ therefore equals

$$
a_n = \frac{1 + \frac{1}{\sqrt{2}}}{\sqrt{2}} (\sqrt{2})^n + \frac{1 - \frac{1}{\sqrt{2}}}{\sqrt{2}} (-\sqrt{2})^n
$$

$$
= \left\{ \begin{array}{ll}
\frac{1+\frac{1}{\sqrt{2}}}{\sqrt{2}} 2^{n/2} + \frac{1-\frac{1}{\sqrt{2}}}{\sqrt{2}} 2^{n/2}, & \text{if } n \text{ is even,} \\
\frac{1+\frac{1}{\sqrt{2}}}{\sqrt{2}} \sqrt{2} 2^{(n-1)/2} - \frac{1-\frac{1}{\sqrt{2}}}{\sqrt{2}} \sqrt{2} 2^{(n-1)/2}, & \text{if } n \text{ is odd,}
\end{array} \right.
$$

$$
= \left\{ \begin{array}{ll}
2^{n/2}, & \text{if } n \text{ is even,} \\
2^{(n-1)/2}, & \text{if } n \text{ is odd,}
\end{array} \right.
$$

$$
= 2^{\left\lfloor n/2 \right\rfloor}.
$$

- It matches Theorem 20 (p. 112).
An Example: A Third-Order Relation

- Consider

\[ 2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \]

with \( a_0 = 0, \ a_1 = 1, \) and \( a_2 = 2. \)

- The characteristic equation

\[ 2r^3 - r^2 - 2r + 1 = 0 \]

has three distinct real roots: 1, \(-1\), and 0.5.

- The general solution is

\[ a_n = c_1 1^n + c_2 (-1)^n + c_3 (1/2)^n \]

\[ = c_1 + c_2 (-1)^n + c_3 (1/2)^n. \]
An Example: A Third-Order Relation (concluded)

- Solving the three initial conditions, we have\(^a\)

\[
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 0.5 \\
1^2 & (-1)^2 & 0.5^2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}.
\]

- The solutions are

\[
c_1 = 2.5, \\
c_2 = 1/6, \\
c_3 = -8/3.
\]

\(^a\)Or see Eq. (79) on p. 545.
The Case of Complex Roots

- Consider

\[ a_n = 2(a_{n-1} - a_{n-2}) \]

with \( a_0 = 1 \) and \( a_1 = 2 \).

- The characteristic equation

\[ r^2 - 2r + 2 = 0 \]

has two distinct complex roots \( 1 \pm i \).

- The general solution is

\[ a_n = c_1(1 + i)^n + c_2(1 - i)^n. \]
The Case of Complex Roots (concluded)

- Solve the two initial conditions for \( c_1 = \frac{(1 - i)}{2} \) and \( c_2 = \frac{(1 + i)}{2} \).

- The particular solution becomes

\[
    a_n = (1 + i)^{n-1} + (1 - i)^{n-1} \\
    = (\sqrt{2})^n [\cos(n\pi/4) + \sin(n\pi/4)].
\]

---

\(^a\)An equivalent one is \( a_n = (\sqrt{2})^{n+1} \cos((n - 1)\pi/4) \) by Mr. Tunglin Wu (B00902040) on May 17, 2012.
$k$th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Repeated Real Roots

- Consider the recurrence relation

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \]

where $C_n, C_{n-1}, \ldots$ are real constants, $C_n \neq 0$, $C_{n-k} \neq 0$.

- Let $r$ be a characteristic root of multiplicity $m$, where $2 \leq m \leq k$, of the characteristic equation

\[ f(x) = C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0. \]

- The general solution that involves $r$ has the form

\[ (A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1}) r^n, \quad (85) \]

with $A_0, A_1, \ldots, A_{m-1}$ are constants to be determined.
The Proof

• If $f(x)$ has a root $r$ of multiplicity $m$, then

$$f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0.$$  

• Because $r \neq 0$ is a root of multiplicity $m$, it is easy to check that

$$0 = r^{n-k}f(r),$$

$$0 = r(r^{n-k}f(r))',$$

$$0 = r(r(r^{n-k}f(r)))',$$

$$\vdots$$

$$0 = r^{m-1}(r^{n-k}f(r))^{(m-1)}'.$$

$$0 = r^{m-1}(r^{n-k}f(r))^{(m-1)}'.$$
The Proof (continued)

• Note that we differentiate and then multiply by $r$ before iterating.

• These give

\[
0 = C_n r^n + C_{n-1} r^{n-1} + \cdots + C_{n-k} r^{n-k},
\]

\[
0 = C_n n r^n + C_{n-1} (n-1) r^{n-1} + \cdots + C_{n-k} (n-k) r^{n-k},
\]

\[
0 = C_n n^2 r^n + C_{n-1} (n-1)^2 r^{n-1} + \cdots + C_{n-k} (n-k)^2 r^{n-k},
\]

\vdots
The Proof (continued)

• Now, \(a_n = n^k r^n\), \(0 \leq k \leq m - 1\), is indeed a solution because the \(k\)th row above says

\[
0 = C_n n^k r^n + C_{n-1} (n-1)^k r^{n-1} + \cdots + C_{n-k} (n-k)^k r^{n-k} \\
= C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k}.
\]
The Proof (continued)

• From Eq. (77) on p. 540, $r^n, nr^n, n^2r^n, \ldots, n^{m-1}r^n$ form a fundamental set if

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 \\
r & r & \cdots & r \\
r^2 & 2r^2 & \cdots & 2^{m-1}r^2 \\
\vdots & \vdots & \ddots & \vdots \\
r^{m-1} & (m-1)r^{m-1} & \cdots & (m-1)^{m-1}r^{m-1}
\end{vmatrix} \neq 0.
\]

• But it is a Vandermonde matrix in disguise.

\(^a\)The \(i\)th row sets \(n = i - 1, i = 1, 2, \ldots, m\).
The Proof (concluded)

• In fact, after deleting the first row and column, the determinant equals

\[(m - 1)! \cdot r^{1+2+\ldots+(m-1)}\]

\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
1 & 2 & \ldots & 2^{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (m - 1) & \ldots & (m - 1)^{m-2}
\end{vmatrix} \neq 0.
\]
Nonhomogeneous Recurrence Relations

• Consider

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \]  (86)

• Suppose \( a_n = a_{n-1} + f(n) \).

• Then the solution is

\[ a_n = a_0 + \sum_{i=1}^{n} f(i). \]

• A closed-form formula exists if one for \( \sum_{i=1}^{n} f(i) \) does.
Nonhomogeneous Recurrence Relations (concluded)

- In general, no failure-free methods exist except for special $f(n)$s.
  - See pp. 441–2 of the textbook (4th ed.).
  - See p. 532 of Rosen (2012) when $f(n)$ is the product of a polynomial in $n$ and the $n$th power of a constant.
Examples \((c, c_1, c_2, \ldots \text{ Are Arbitrary Constants})\)

<table>
<thead>
<tr>
<th>(a_{n+1} - a_n = 0)</th>
<th>(a_n = c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{n+1} - a_n = 1)</td>
<td>(a_n = n + c)</td>
</tr>
<tr>
<td>(a_{n+1} - a_n = n)</td>
<td>(a_n = n(n - 1)/2 + c)</td>
</tr>
<tr>
<td>(a_{n+2} - 3a_{n+1} + 2a_n = 0)</td>
<td>(a_n = c_1 + c_22^n)</td>
</tr>
<tr>
<td>(a_{n+2} - 3a_{n+1} + 2a_n = 1)</td>
<td>(a_n = c_1 + c_22^n - n)</td>
</tr>
<tr>
<td>(a_{n+2} - a_n = 0)</td>
<td>(a_n = c_1 + c_2(-1)^n)</td>
</tr>
<tr>
<td>(a_{n+1} = a_n/(1 + a_n))</td>
<td>(a_n = c/(1 + cn))</td>
</tr>
</tbody>
</table>
Trial and Error

- Consider $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- Calculations show that $a_2 = 4$ and $a_3 = 12$.
- Conjecture:
  \[ a_n = n2^{n-1}. \]  
  \( (87) \)
- Verify that, indeed,
  \[ (n + 1)2^n = 2(n2^{n-1}) + 2^n, \]
  and $a_1 = 1$. 
Application: Number of Edges of a Hasse Diagram

• Let $a_n$ be the number of edges of the Hasse diagram for the partial order $(2\{1,2,\ldots,n\}, \subseteq)$.

• Consider the Hasse diagrams $H_1$ for $(2\{1,2,\ldots,n\}, \subseteq)$ and $H_2$ for $(\{ T \cup \{ n+1 \} : T \subseteq \{ 1, 2, \ldots, n \} \}, \subseteq)$.
  - $H_1$ and $H_2$ are “isomorphic.”

• The Hasse diagram for $(2\{1,2,\ldots,n+1\}, \subseteq)$ is constructed by adding an edge from each node $T$ of $H_1$ to node $T \cup \{ n+1 \}$ of $H_2$.

• Hence $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.

• The desired number has been solved in Eq. (87) on p. 586.
Illustration with \((2^{\{1,2,3\}}, \subseteq)\)
Trial and Error Again

• Consider $a_{n+1} - Aa_n = B$.

• Calculations show that
  
  \[
  a_1 = Aa_0 + B, \\
  a_2 = Aa_1 + B = A^2a_0 + B(A + 1), \\
  a_3 = Aa_2 + B = A^3a_0 + B(A^2 + A + 1).
  \]

• Conjecture (easily verified by substitution):
  
  \[
  a_n = \begin{cases} 
  A^n a_0 + B \frac{A^n - 1}{A - 1}, & \text{if } A \neq 1 \\
  a_0 + Bn, & \text{if } A = 1
  \end{cases} \quad (88)
  \]
Financial Application: Compound Interest\textsuperscript{a}

• Consider $a_{n+1} = (1 + r) a_n$.
  – Deposit grows at a period interest rate of $r > 0$.
  – The initial deposit is $a_0$ dollars.

• The solution is obviously

$$a_n = (1 + r)^n a_0.$$ 

• The deposit therefore grows exponentially with time.

\textsuperscript{a}“In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect,” wrote Joseph Alois Schumpeter (1883–1950) in \textit{Capitalism, Socialism and Democracy} (1942).
Financial Application: Amortization

- Consider \( a_{n+1} = (1 + r) a_n - M \).
  - The initial loan amount is \( a_0 \) dollars.
  - The monthly payment is \( M \) dollars.
  - The outstanding loan principal after the \( n \)th payment is \( a_n \).

- By Eq. (88) on p. 589, the solution is
  \[
  a_n = (1 + r)^n a_0 - M \frac{(1 + r)^n - 1}{r}.
  \]
The Proof (concluded)

- What is the unique monthly payment \( M \) for the loan to be closed after \( k \) monthly payments?

- Set \( a_k = 0 \) to obtain

\[
\begin{align*}
    a_k &= (1 + r)^k a_0 - M \frac{(1 + r)^k - 1}{r} = 0.
\end{align*}
\]

- Hence

\[
    M = \frac{(1 + r)^k a_0 r}{(1 + r)^k - 1}.
\]

- This is a standard formula for home mortgages and annuities.\(^a\)

\(^a\)Lyuu (2002).
Trial and Error a Third Time

• Consider the more general \( a_{n+1} - Aa_n = BC^n \).

• Calculations show that

\[
\begin{align*}
a_1 &= Aa_0 + B, \\
a_2 &= Aa_1 + BC = A^2a_0 + B(A + C), \\
a_3 &= Aa_2 + BC^2 = A^3a_0 + B(A^2 + AC + C^2).
\end{align*}
\]

• Conjecture (easily verified by substitution):

\[
a_n = \begin{cases} 
  A^n a_0 + B \frac{A^n - C^n}{A - C}, & \text{if } A \neq C \\
  A^n a_0 + BA^{n-1}n, & \text{if } A = C
\end{cases}.
\] (89)
Application: Runs of Binary Strings

- A run is a maximal consecutive list of identical objects (p. 114).
  - Binary string “0 0 1 1 1 0” has 3 runs.
- Let $r_n$ denote the total number of runs determined by the $2^n$ binary strings of length $n$.
- First, $r_1 = 2$.
  - Each of “0” and “1” has 1 run.
- Next, $r_2 = 6$.
  - “00” and “11” each has 1 run, while “01” and “10” each has 2 runs.
The Proof (continued)

• In general, suppose we append a bit to every \((n - 1)\)-bit string \(b_1 b_2 \cdots b_{n-1}\) to make \(b_1 b_2 \cdots b_{n-1} b_n\).

• First, suppose \(b_{n-1} = b_n\) (i.e., the last 2 bits are identical).

• Then the total number of runs does not change.
  – The total number of runs remains \(r_{n-1}\).
The Proof (continued)

• Next, suppose $b_{n-1} \neq b_n$ (i.e., the last 2 bits are distinct).

• Then the total number of runs increases by 1 for each $(n - 1)$-bit string.
  – There are $2^{n-1}$ of them.
  – So the total number of runs becomes $r_{n-1} + 2^{n-1}$. 
The Proof (continued)

- Hence

\[ r_n = 2r_{n-1} + 2^{n-1}, \quad n \geq 2. \]  

(90)

- By Eq. (89) on p. 593,

\[ r_n = 2^n r_0 + 2^{n-1} n. \]

- To make sure that \( r_1 = 2 \), it is easy to see that \( r_0 = 1/2 \).

- Hence

\[ r_n = 2^{n-1} + 2^{n-1} n = 2^{n-1} (n + 1). \]
The Proof (concluded)

• The recurrence (90) is identical to that for the number of edges of a Hasse diagram (p. 587).

• But the initial condition was different: $a_1 = 1$.

• Its slightly different solution appeared in Eq. (87) on p. 586: $a_n = n2^{n-1}$. 
Method of Undetermined Coefficients

• Recall Eq. (86) on p. 583, repeated below:

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \quad (91) \]

• Let \( a_n^{(h)} \) denote the general solution of the associated homogeneous relation (with \( f(n) = 0 \)).

• Let \( a_n^{(p)} \) denote a particular solution of the nonhomogeneous relation.

• Then

\[ a_n = a_n^{(h)} + a_n^{(p)}. \]

• All the entries in the table on p. 585 fit the claim.
Conditions for the General Solution

Similar to Theorem 69 (p. 540), we have the following theorem.

**Theorem 70** Let $a_n^{(p)}$ be any particular solution of the nonhomogeneous recurrence relation Eq. (91) on p. 599. Let

$$a_n^{(h)} = C_1 a_n^{(1)} + C_2 a_n^{(2)} + \cdots + C_k a_n^{(k)}$$

be the general solution of its homogeneous version as specified in Theorem 69. Then $a_n^{(h)} + a_n^{(p)}$ is the general solution of Eq. (91) on p. 599.
Solution Techniques

• Typically, one finds the general solution of its homogeneous version $a_n^{(h)}$ first.

• Then one finds a particular solution $a_n^{(p)}$ of the nonhomogeneous recurrence relation Eq. (91) on p. 599.

• Make sure $a_n^{(p)}$ is “independent” of $a_n^{(h)}$.

• Finally, use the initial conditions to nail down the coefficients of $a_n^{(h)}$.

• Output $a_n^{(h)} + a_n^{(p)}$. 
\[ a_{n+1} - Aa_n = B \] Revisited

- Recall that the general solution is \( a_n^{(h)} = cA^n \).
- A particular solution is (verify it)
  \[
  a_n^{(p)} = \begin{cases} 
  B/(1 - A), & \text{if } A \neq 1 \\
  Bn, & \text{if } A = 1
  \end{cases}
  \]
- So \( a_n = cA^n + a_n^{(p)} \).
- In particular,
  \[
  c = a_0 - a_0^{(p)} = \begin{cases} 
  a_0 - B/(1 - A), & \text{if } A \neq 1 \\
  a_0, & \text{if } A = 1
  \end{cases}
  \]
\[ a_{n+1} - Aa_n = B \]

Revisited (concluded)

- The solution matches Eq. (88) on p. 589.
- We can rewrite the solution as

\[
a_n = \begin{cases} 
A^n [a_0 - a_n^{(p)}] + a_n^{(p)}, & \text{if } A \neq 1 \\
 a_0 + a_n^{(p)}, & \text{if } A = 1 
\end{cases}
\]  

\text{(92)}
Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 7^n$ with $a_0 = 2$

- $a_n^{(h)} = c \times 3^n$, because the characteristic equation has the nonzero root 3.

- We propose $a_n^{(p)} = a \times 7^n$.

- Place $a \times 7^n$ into the relation to obtain $a \times 7^n - 3a \times 7^{n-1} = 5 \times 7^n$.

- Hence $a = 35/4$ and $a_n^{(p)} = (35/4) \times 7^n = (5/4) \times 7^{n+1}$.

- The general solution is $a_n = c \times 3^n + (5/4) \times 7^{n+1}$.

- Now, $c = -27/4$ because $a_0 = 2 = c + (5/4) \times 7$.

- So the solution is $a_n = -(27/4) \times 3^n + (5/4) \times 7^{n+1}$. 
Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 3^n$ with $a_0 = 2$

- As before, $a_n^{(h)} = c \times 3^n$.
- But this time $a_n^{(h)}$ and $f(n) = 5 \times 3^n$ are not "independent."
- So propose $a_n^{(p)} = an \times 3^n$.
- Plug $an \times 3^n$ into the relation to obtain $an \times 3^n - 3a(n - 1) \times 3^{n-1} = 5 \times 3^n$.
- Hence $a = 5$ and $a_n^{(p)} = 5n \times 3^n$.
- The general solution is $a_n = c \times 3^n + 5n \times 3^n$.
- Finally, $c = 2$ with use of $a_0 = 2$. 

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Nonhomogeneous $a_{n+1} - 2a_n = n + 1$ with $a_0 = 4$

- From Eq. (88) on p. 589, $a_n^{(h)} = c \times 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute this particular solution into the relation to yield
  $$a(n + 1) + b - 2(an + b) = n + 1.$$  
- Rearrange the above to obtain
  $$(-a - 1)n + (a - b - 1) = 0.$$  
- This holds for all $n$ if $a = -1$ and $b = -2$. 
The Proof (concluded)

• Hence \( a_n^{(p)} = -n - 2 \).
• The general solution is
  \[ a_n = c \times 2^n - n - 2. \]
• Use the initial condition
  \[ 4 = a_0 = c - 2 \]
  to obtain \( c = 6 \).
• The solution to the complete relation is
  \[ a_n = 6 \times 2^n - n - 2. \]
Nonhomogeneous $a_{n+1} - a_n = 2n + 3$ with $a_0 = 1$

• This equation is very similar to the previous one:

$$a_{n+1} - 2a_n = n + 1.$$ 

• First, $a_n^{(h)} = d \times 1^n = d$.

• If one guesses $a_n^{(p)} = an + b$ as before, then

$$a_{n+1} - a_n = a(n + 1) + b - an - b = a,$$

which cannot be right.\(^a\)

• So we guess $a_n^{(p)} = an^2 + bn + c$.

\(^a\)Contributed by Mr. Yen-Chieh Sung (B01902011) on June 17, 2013.
The Proof (continued)

- Substitute this particular solution into the relation to yield
  \[ a(n + 1)^2 + b(n + 1) + c - (an^2 + bn + c) = 2n + 3. \]

- Simplify the above to obtain
  \[ 2an + (a + b) = 2n + 3. \]

- The solutions are \( a = 1 \) and \( b = 2 \).

- Hence \( a_n^{(p)} = n^2 + 2n + c \).

- The general solution is \( a_n = n^2 + 2n + c. \)

\[ ^{a}\text{We merge } d \text{ into } c. \]
The Proof (concluded)

• Use the initial condition

\[ 1 = a_0 = c \]

to obtain \( c = 1 \).

• The solution to the complete relation is

\[ a_n = n^2 + 2n + 1 = (n + 1)^2. \]

• It is very different from the solution to the previous example:

\[ a_n = 6 \times 2^n - n - 2. \]
Nonhomogeneous $a_{n+2} - 3a_{n+1} + 2a_n = 2$ with $a_0 = 0$ and $a_1 = 2$

- The characteristic equation $r^2 - 3r + 2 = 0$ has roots 2 and 1.
- So $a_n^{(h)} = c_1 1^n + c_2 2^n = c_1 + c_2 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute $a_n^{(p)}$ into the relation to yield
  
  $$a(n + 2) + b - 3[a(n + 1) + b] + 2(an + b) = 2.$$ 

- Rearrange the above to obtain $a = -2$.
- Hence $a_n^{(p)} = -2n + b$. 
The Proof (concluded)

• The general solution is now \( a_n = c_1 + c_2 2^n - 2n \).\(^a\)

• Use the initial conditions

\[
0 = a_0 = c_1 + c_2, \\
2 = a_1 = c_1 + 2c_2 - 2.
\]

To obtain \( c_1 = -4 \) and \( c_2 = 4 \).

• The solution to the complete relation is

\[
a_n = -4 + 2^{n+2} - 2n.
\]

\(^a\)We merge \( b \) into \( c_1 \).
The Method of Generating Functions (Recall p. 558)

- Consider the relation \( a_n - 3a_{n-1} = n \) with \( a_0 = 1 \).
- Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be the generating function for \( a_0, a_1, \ldots \).
- From the recurrence equation,
  \[
  \sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n.
  \]
- \( f(x) - a_0 - 3xf(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \) from p. 468.
- Hence
  \[
  f(x) = \frac{x}{(1-x)^2} + 1.
  \]
The Method of Generating Functions (continued)

• Now,

\[ f(x) = \frac{1}{1 - 3x} + \frac{x}{(1 - x)^2(1 - 3x)} = \frac{7/4}{1 - 3x} + \frac{-1/4}{1 - x} + \frac{-1/2}{(1 - x)^2} \]

by a \textbf{partial fraction decomposition}.

– The following equivalent form is \textit{not} a partial fraction decomposition:

\[ \frac{7/4}{-3x + 1} + \frac{x - 3}{(1 - x)^2}. \]
The Method of Generating Functions (continued)

• Now,

\[
\frac{7/4}{1 - 3x} = \left(\frac{7/4}{1 - 3x}\right) \sum_{n=0}^{\infty} (3x)^n,
\]

\[
-\frac{1/4}{1 - x} = -\left(\frac{1/4}{1 - x}\right) \sum_{n=0}^{\infty} x^n,
\]

\[
-\frac{1/2}{(1 - x)^2} = -\left(\frac{1/2}{(1 - x)^2}\right) \sum_{n=0}^{\infty} (n + 1) x^n,
\]

from p. 467.
The Method of Generating Functions (concluded)

• Now,

\[
f(x) = \left(\frac{7}{4}\right) \sum_{n=0}^{\infty} 3^n x^n - \left(\frac{1}{4}\right) \sum_{n=0}^{\infty} x^n - \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} (n + 1) x^n.
\]

• So

\[a_n = \left(\frac{7}{4}\right) 3^n - \left(\frac{1}{4}\right) - \left(\frac{1}{2}\right)(n + 1).
\]

• The methodology should be clear.