Existence of Nodes with Identical Degree

- Let $G = (V,E)$ be a loop-free connected undirected graph with $n = |V| \geq 2$.
- Observe that $1 \leq \deg(v) \leq n - 1$.
- But there are $n$ nodes.
- By the pigeonhole principle (p. 307), there must be 2 nodes with the same degree.
Regular Graphs

• A \textit{d-regular graph} is an undirected graph such that every node has degree \( d \).

• An \( d \)-regular graph \( G = (V, E) \) must have an even number of nodes if \( d \) is odd.
  - By Eq. (91) on p. 615, \( 2 \times |E| = d \times |V| \).
  - As \( d \) is odd, \( |V| \) must be even.
The Hypercube

• The nodes of the $n$-dimensional hypercube $Q_n$ are represented as $n$-bit numbers (see p. 555).
  – There are $2^n$ nodes.

• Two nodes are connected if they differ in one dimension.
  – For example, there is an edge between 00100 and 00110.
  – The diameter is $n$.
  – It is $n$-regular.
  – There are
    \[
    \frac{n2^n}{2} = n2^{n-1}
    \]
    undirected edges.
The Hypercube (concluded)

• The hypercube was once a popular topology for massively parallel processors (MPPs).

• The record is $n = 16$ set by Thinking Machine Corp.’s Connection Machine CM-2.\(^a\)

\(^a\)Hillis (1985).
Illustration with $Q_3$
Bipartite Graphs

• A graph $G = (V, E)$ is called bipartite if:
  - $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
  - Every edge is of the form $\{x, y\}$ with $x \in V_1$ and $y \in V_2$.

• Express the above bipartite graph as
  $$G = (V_1, V_2, E).$$

• If each node in $V_1$ is joined with every node in $V_2$, we have a complete bipartite graph.
  - If $|V_1| = m$ and $|V_2| = n$, the complete bipartite graph is denoted by $K_{m,n}$. 
$K_{5,5}$
Bipartite Graphs (concluded)

• Let graph $G = (V, E) = (V_1, V_2, E)$ be bipartite.
• Then $G$ has at most $|V_1| \times |V_2|$ edges.
• Let $|V| = n$, $|E| = e$, and $|V_1| = m$.
• Then $e \leq (n - m) m$, which is maximized at (1) $m = n/2$ when $n$ is even and (2) $m = (n \pm 1)/2$ when $n$ is odd.
• In either case,
  
  $e \leq (n/2)^2$.
• Hence a graph with $e > (n/2)^2$ cannot be bipartite.
Euler Circuits and Trails\textsuperscript{a}

- Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes.
  - Isolated nodes are nodes without incident edges.

- $G$ is said to have an Euler circuit if there is a circuit in $G$ that traverses every edge of the graph exactly once.
  - You can draw the edges without lifting the pen.

- If there is an open trail from $x$ to $y$ in $G$ and this trail traverses every edge of the graph exactly once, the trail is called an Euler trail.

\textsuperscript{a}Euler in 1736, the year graph theory was born.
Characterization of Having Euler Circuits

**Theorem 73 (Euler (1736))** Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes. Then $G$ has an Euler circuit if and only if $G$ is connected and every node in $G$ has an even degree.

- Testing if a graph is Eulerian hence is trivial.
- The proof will be constructive.
- Let $n = |E|$. 
The Proof ($\Rightarrow$)

- Clearly $G$ is connected.
- Each time the Euler circuit enters a non-starting node $v$, it must exit it before coming back again, if ever.
- This contributes a count of 2 to $\deg(v)$.
- Because every edge is traversed, $\deg(v)$ must be even.
- The Euler circuit must start from the starting node $s$ and end at the same starting node.
- Each exit is matched by one entry.
- So $\deg(s)$ is even.
The Proof ($\Leftarrow$)

- The $n = 1, 2$ cases are easy, by inspection.
- Assume the result is true when there are $< n$ edges.
- If $G$ has $n$ edges, select a node $s \in G$ as the starting and ending node.
- Construct a circuit $C$ from $s$.
  - Start from $s$.
  - Traverse any hitherto untraversed edge, and so on.
  - We must eventually return to $s$ because every node has an even degree and hence the last visit to it must be an exit, except $s$. 
The Proof ($\iff$) (continued)

- If $C$ traverses every edge, we are done.
- Otherwise, remove the edges of $C$ and isolated nodes to yield a new graph $K$.
- The degree of each node in $K$ remains even.
The Proof ($\iff$) (continued)$^a$

- Suppose $K$ is connected and $s$ is not isolated
  - Construct an Euler circuit $c$ of $K$ (doable by the induction hypothesis).
  - Node $s$ is on this Euler circuit because $s \in K$ and $K$ is connected.
  - The desired Euler circuit: Start from $s$ and travel on $C$ until we end at $s$ and then traverse $c$ until we end at $s$ again.

$^a$With input from Mr. Cheng-Yu Lee (B91902103) on December 1, 2003.
The Proof ($\Leftarrow$) (concluded)

- Suppose $K$ is disconnected or $s$ is isolated.
  - Construct an Euler circuit $c_i$ in each component of $K$
    (doable by the induction hypothesis).
  - Each component must have at least one node in common with $C$
    because originally $G$ is connected.
  - Let $s_i$ be the first node with which $C$ visits $c_i$.\(^{\text{a}}\)
  - The desired Euler circuit: Start from $s$ and travel on $C$
    until we reach $s_1$, traverse $c_1$, return to $s_1$, continue on $C$
    until we reach $s_2$, and so on.

\(^{\text{a}}C\) may visit many nodes of $c_i$. Thanks to a lively class discussion on
Constructing an Euler Circuit
Characterization of Having Euler Trails

**Corollary 74** Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes. Then $G$ has an Euler trail if and only if $G$ is connected and has exactly two nodes of odd degree.

- Let $x, y$ be the two nodes of odd degree.
- Add edge $\{x, y\}$ to $G$.
- Construct an Euler circuit, which exists by Theorem 73.
- Remove the edge $\{x, y\}$ from the circuit to arrive at an Euler trail.
In and Out Degrees

- Let $G$ be a directed graph.
- The **in degree** of $v \in V$ is the number of edges in $G$ that are incident into $v$.
- The **out degree** of $v \in V$ is the number of edges in $G$ that are incident from $v$.
  - The in and out degrees of a node may not equal.
- Similar to the definition of (undirected) regular graphs (p. 617), a directed $d$-regular graph is a directed graph such that every node has in-degree and out-degree $d$. 
Characterization of Having Directed Euler Circuits

Theorem 75  Let $G = (V, E)$ be a digraph. Then $G$ has a directed Euler circuit if and only if $G$ is connected and the in degree equals the out degree at every node.

- Follow the same proof as Theorem 73 (p. 625).
- The only difference is that, whereas we maintained even node degrees, we now maintain the equality of in and out degrees.
Euler Circuits: Additional Remarks

- Counting the number of Euler circuits for digraphs can be solved efficiently.\(^\text{b}\)

- Counting the number of Euler circuits for undirected graphs is computationally hard—it is \#P-complete.\(^\text{c}\)

- Asymptotic formulas exist for the number of Euler circuits on \(K_n\) when \(n\) is odd.\(^\text{d}\)

\(^\text{a}\)Contributed by Mr. Eric Ruei-Min Lee (B00902106) on June 4, 2012.

\(^\text{b}\)Harary and Palmer (1973).

\(^\text{c}\)Brightwell and Winkler (2004).

\(^\text{d}\)McKay and Robinson (1995).
Planar Graphs

- A graph or multigraph $G$ is called planar if it can be drawn in the plane with the edges intersecting only at nodes of $G$.

- Planarity can be tested efficiently.\textsuperscript{a}

\textsuperscript{a}Hopcroft and Tarjan (1974).
A Planar Graph

Such a drawing of $G$ is called an embedding of $G$ in the plane.
Euler’s Theorem\textsuperscript{a}

- Let $G = (V, E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$.
- Let $r$ be the number of regions in the plane determined by a planar embedding of $G$.
- One of these regions has infinite area and is called the infinite region.
- Then

$$v - e + r = 2.$$  \hfill (92)

\textsuperscript{a}Euler (1752).
A Planar Graph with $v = 16$, $e = 35$, $r = 21$
The Proof\textsuperscript{a}

- The theorem holds if $e = 0, 1$.\textsuperscript{b}

- Assume the theorem holds for any connected planar graph with $e$ edges, where $0 \leq e \leq k$.

- Let $G = (V, E)$ be a connected planar graph with $v$ nodes, $r$ regions, and $e = k + 1$ edges.

- Let $\{x, y\} \in E$.

- Delete $\{x, y\}$ to obtain graph $H$: $G = H + \{x, y\}$.

\textsuperscript{a}See Imre Lakatos (1922–1974), Proofs and Refutations: The Logic of Mathematical Discovery (1989), for a most penetrating presentation.

\textsuperscript{b}See p. 545 of the textbook (5th ed.).
The Proof When $H$ Is Connected

- The dotted edge on p. 642 is $\{x, y\}$.
- So $H$ has $v$ nodes, $k$ edges, and $r - 1$ regions.
- $H$ is also planar.
- The induction hypothesis applied to $H$ says
  \[ v - k + (r - 1) = 2. \]
- Hence
  \[ v - (k + 1) + r = 2. \]
- The theorem is proved because $G$ has $v$ nodes, $e = k + 1$ edges, and $r$ regions.
A Planar $G$ from a Planar $H$
The Proof When $H$ Is Not Connected

- The dotted edge on p. 644 is $\{x, y\}$.
- So $H$ has $v$ nodes, $k = e - 1$ edges, and $r$ regions.
- $H$ has two components $H_1$ and $H_2$, both planar.
- Let $H_i$ have $v_i$ nodes, $e_i$ edges, and $r_i$ regions.
- The induction hypothesis applied to $H_i$ says
  \[v_i - e_i + r_i = 2.\]
- Therefore,
  \[ (v_1 + v_2) - (e_1 + e_2) + (r_1 + r_2) = 4. \] (93)

\(^a\)Thanks to a lively class discussion on December 1, 2003.
A Planar $G$ from Planar $H_1$ and $H_2$
The Proof When $H$ Is Not Connected (concluded)

- Now,

\begin{align*}
v_1 + v_2 &= v, \\
e_1 + e_2 &= k = e - 1, \\
r_1 + r_2 &= r + 1.
\end{align*}

- Hence Eq. (93) on p. 643 becomes

\[ v - (e - 1) + (r + 1) = 4. \]

- So, again, $v - e + r = 2$. 
A Useful Corollary

**Corollary 76** Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$ and $|E| = e > 2$. Then

$$e \leq 3v - 6.$$

- Let there be $r$ regions.
- Each edge is shared by $\leq 2$ regions.
- The boundary of each region (including the infinite region) contains at least 3 edges ($G$ is not a multigraph).
- Hence $2e \geq \sum_{\text{region } R} |R\text{'s boundary}| \geq 3r$.
- Euler’s theorem implies

$$2 = v - e + r \leq v - e + (2/3)e = v - (1/3)e.$$
$K_5$ Is Not Planar

• $K_5$ has $v = 5$ nodes and $e = 10$ edges.
• Suppose it is planar.
• By Corollary 76,

\[
10 = e \leq 3v - 6 = 9,
\]

a contradiction.
$K_{3,3}$ Is Not Planar

- $K_{3,3}$ has $v = 6$ nodes and $e = 9$ edges.
- Suppose it is planar.
- By Euler’s formula (92) on p. 638, the number of regions is

$$r = 2 + e - v = 5.$$
The Proof (concluded)

- But $K_{3,3}$ has no 3 nodes forming a complete subgraph.
- So the border of a region must contain at least 4 edges.
- The sum of those edges is at least $4r = 20$.
- By Eq. (91) on p. 615,
  \[
  \sum_{v \in V} \deg(v) = 2e = \sum_{\text{region } R} |R's \text{ boundary}| \geq 20,
  \]
  contradicting $e = 9$. 

Kuratowski’s\textsuperscript{a} Theorem

Theorem 77 (Kuratowski (1930)) A graph is nonplanar if and only if it contains a subgraph that is “homeomorphic” to either $K_5$ or $K_{3,3}$.

Corollary 78 (1) Shrinking any edge of a planar graph to a single node preserves planarity. (2) Shrinking any connected component of a planar graph to a single node preserves planarity.

\textsuperscript{a}Kasimir Kuratowski (1896–1980).
Hamiltonian\textsuperscript{a} Paths and Cycles

\begin{itemize}
  \item Let \( G = (V, E) \) be a graph with \(|V| \geq 3\).
  
  \item A \textbf{Hamiltonian cycle} is a \textit{cycle} in \( G \) that contains every node (exactly once) in \( V \).
  
  \item A \textbf{Hamiltonian path} is a \textit{path} in \( G \) that contains every node (exactly once) in \( V \).
  
  \item Testing if \( G \) has a Hamiltonian path or cycle is computationally hard—it is NP-complete.\textsuperscript{b}
\end{itemize}

\textsuperscript{a}William Rowan Hamilton (1805–1865).
\textsuperscript{b}Karp (1972).
William Rowan Hamilton (1805–1865)
Richard Karp\textsuperscript{a} (1935–)

\textsuperscript{a}Turing Award (1985).
Application: Tournaments

- Let $K_n^*$ be a directed graph with $n$ nodes.

- If for each distinct pair $x, y$ of nodes, either $(x, y) \in K_n^*$ or $(y, x) \in K_n^*$ but not both, then $K_n^*$ is called a tournament.\(^a\)

- A tournament is not necessarily transitive.
  - A digraph $(V, E)$ is transitive if
    \[
    (a, b) \in E \land (b, c) \in E \Rightarrow (a, c) \in E.
    \]

- But the next theorem says that players can be ranked in at least one way.

\(^a\)Recall p. 316.
Theorem 79 (Redei (1934)) A tournament always contains a directed Hamiltonian path.

- Let $p_m = (v_1, v_2, \ldots, v_m)$ be a path of maximum length.
- Assume $m < n = |V|$ and proceed to derive a contradiction.
- Let $v$ be a node not on $p_m$.
- $(v, v_1) \notin K^*_n$ for otherwise $p_m$ can be lengthened to $(v, v_1, v_2, \ldots, v_m)$.
- Hence $(v_1, v) \in K^*_n$.

\[\text{\textsuperscript{a}}\text{Similar results appear on p. 318 and p. 353.}\]
The Proof (continued)

- If there exists a $1 < j \leq m$ such that $(v_{j-1}, v) \in K^*_n$ and $(v, v_j) \in K^*_n$, then the path $(v_1, \ldots, v_{j-1}, v, v_j, \ldots, v_m)$ is longer than $p_m$, a contradiction.\(^a\)

- As $(v_1, v) \in K^*_n$, we conclude that for each $1 < j \leq m$, $(v_{j-1}, v) \in K^*_n$ and $(v, v_j) \not\in K^*_n$ by induction.

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\(^a\)Improved by a lively discussion on June 5, 2014.
The Proof (concluded)

- In particular, \((v, v_m) \notin K_n^*,\) so \((v_m, v) \in K_n^*.\)
- We can add \((v_m, v)\) to \(p_m\) to make it longer, a contradiction.
- Remark: Now that \(K_n^*\) is Hamiltonian, how to find a Hamiltonian path efficiently?
Graph Coloring

• Let $G = (V, E)$ be an undirected graph.

• A proper coloring of $G$ occurs when its nodes are colored so that adjacent nodes have different colors.

• The minimum number of colors needed to color $G$ is the chromatic number of $G$ and is written as $\chi(G)$.

• Four colors suffice to color any planar graph.\(^a\)

\(^a\)Kenneth Appel and Wolfgang Haken (1976). Although the original proof uses a computer, a computer-generated formal proof has been given by Gonthier (2004)! This theorem was examined in 1850 by Francis Guthrie (1831–1899) and made its official birth in a letter from De-Morgan to Hamilton in 1852. Kenneth Appel (1932–2013), “Without computers, we would be stuck only proving theorems that have short proofs.”
Graph Coloring (concluded)

- The graph colorability problem for 3 colors and up is computationally hard—it is NP-complete.$^a$

- The number of ways to color a graph on $n$ nodes using $k$ colors can be calculated in time $O(2^n n^{O(1)}).$ $^b$

- $\chi(G)$ can be calculated in time $O(2.2461^n).$ $^c$

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$^a$ Karp (1972).

$^b$ Bjöklund and Husfeldt (2006).

$^c$ Bjöklund and Husfeldt (2006).
The four-color theorem says that any map can be colored with just 4 colors.
Elementary Facts

• For all $n \geq 1$, $\chi(K_n) = n$.
  - Each node is adjacent to $n - 1$ other nodes.

• If $H$ is a subgraph of $G$, then $\chi(H) \leq \chi(G)$.
  - A proper coloring of $G$ is also one of $H$.

• An undirected graph $G$ is bipartite if and only if $\chi(G) \leq 2$.
  - Given a bipartite partition $V = V_1 \cup V_2$, color $V_1$ and $V_2$ with two different colors.
An Upper Bound on the Chromatic Number

Theorem 80 (Vizing (1964)) Every graph is $(\kappa + 1)$-colorable, where $\kappa$ is the maximum degree of the nodes.

1: while $G(V, E)$ has uncolored nodes do
2: Pick an arbitrary uncolored $v \in V$;
3: Choose color $c$ that is not used by $v$’s $\leq \kappa$ neighbors;
4: Color $v$ with $c$;
5: end while
Comments on Vizing’s Theorem

• This bound is tight because $\chi(K_n) = n$ (p. 662).

• The neighbors may be colored with the same colors, so $\kappa + 1$ is not a lower bound for the chromatic number.\(^a\)

\(^a\)Contributed by Mr. Asger K. Pedersen (T02202107) on June 5, 2014.
Coloring 3-Colorable Graphs Efficiently

Theorem 81 (Wigderson (1983)) Any 3-colorable graph can be colored in polynomial time with $O(\sqrt{n})$ colors.

- Surprisingly, no one knows how to do better!\(^a\)

\(^a\)Williamson and Shmoys (2011).
Trees
I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

Most mathematicians work with calculus-type “smooth” problems, not discrete things like cleverly arranged arrays of zeros and ones.
Trees

- A **tree** is a loop-free undirected graph that is connected and contains no cycles.

- A **forest** is a loop-free undirected graph whose components are trees.

- A **spanning tree** for a connected graph \( G = (V, E) \) is a subgraph of \( G \) with the same node set \( V \) that is also a tree.\(^a\)
  
  - A spanning tree is computationally easy to construct.

**Lemma 82** A loop-free connected undirected graph has cycles if and only if it is not a tree.

- By definition of tree.

\(^a\)Borůvka (1926).
A Tree
A Spanning Tree

The solid lines constitute the edges of a spanning tree.

An undirected graph has a spanning tree if and only if it is connected.
Properties of Trees

- If $x$ and $y$ are distinct nodes in a tree, then there is a unique path that connects them.
  - There is at least one such path because a tree is connected.
  - But more than one such path implies the existence of a cycle, a contradiction.
Properties of Trees (continued)

**Theorem 83** *For a tree $(V, E)$, $|V| = |E| + 1$.*

- Obviously true when $|E| = 0$ as it is a single node.
- In general, a tree with $|E| = k + 1$ edges breaks into two trees $(V_1, E_1)$ and $(V_2, E_2)$ by the deletion of an edge.
- By the induction hypothesis, $|V_1| = |E_1| + 1$ and $|V_2| = |E_2| + 1$.
- Hence,

\[ |V| = |V_1| + |V_2| = |E_1| + |E_2| + 2 = |E| + 1. \]
Properties of Trees (continued)

- Theorem 82 (p. 672) may hold for nontrees.
- Consider the following graph:
- It satisfies Theorem 82.
- But the graph is not connected.
Properties of Trees (concluded)

The following statements are equivalent for a loop-free undirected graph $G = (V, E)$.

1. $G$ is a tree.

2. $G$ is connected, but the removal of any edge disconnects $G$ into two subgraphs that are trees.


4. $G$ is connected, and $|V| = |E| + 1$.

5. $G$ contains no cycles, and if $x, y \in V$ with $\{x, y\} \notin E$, then the graph obtained by adding edge $\{x, y\}$ to $G$ has precisely one cycle.
Trees and Forests

Corollary 84  For a forest \((V, E)\), \(|V| = |E| + \kappa\), where \(\kappa\) is the number of trees in the forest.

- From Theorem 82 (p. 672), \(|V_i| = |E_i| + 1\) for each tree in the forest.
- Hence

\[
|V| = \sum_{i=1}^{\kappa} |V_i| = \sum_{i=1}^{\kappa} (|E_i| + 1) = |E| + \kappa.
\]
Trees and Cycles

Corollary 85 If a loop-free connected undirected graph is not a tree, then $|V| < |E| + 1$.

- Suppose $|V| \geq |E| + 1$ instead.
- Because the graph is not a tree, $|V| > |E| + 1$ by Property 4 on p. 674.
- But then the graph cannot be connected (why?), a contradiction.
Trees Have the Most Nodes

**Corollary 86** Among loop-free connected undirected graphs, trees have more nodes than nontrees.

- Consider a graph with $m$ edges.
- From Corollary 85 (p. 676), a nontree must have $\leq m$ nodes.
- But Theorem 82 (p. 672) says that a tree with $m$ edges has $m + 1$ nodes.
Coloring of Trees

Theorem 87  Every tree is 2-colorable.

- Pick any node \( v \).
- Color any node reachable from \( v \) via an odd number of edges red.
- Color any node reachable from \( v \) via an even number of edges blue.
- Because a tree has no cycles (Lemma 84 on p. 676), the above operations will not contradict each other at any node.
Planarity of Trees

Lemma 88  Trees are planar.

- A tree contains no cycles.
- So it cannot contain a subgraph homeomorphic to either $K_{3,3}$ or $K_5$.
- The lemma follows by Kuratowski’s theorem (p. 650).
Theorem 82 (p. 672) Reproved

- As a tree $(V, E)$ is planar by Lemma 88 (p. 679), Euler’s theorem (p. 638) says $|V| - |E| + 1 = 2$.

- But this is exactly what Theorem 82 says,

  \[ |V| = |E| + 1. \]
Rooted Trees

• Let $G$ be a directed graph.

• $G$ is called a **directed tree** if the undirected graph associated with $G$ is a tree.

• A directed tree $G$ is called a **rooted tree** if there is a unique node $r$, called the root, with an in degree of zero and for all other nodes $v$, the in degree of $v$ is 1.

• A node with an out degree of zero is called a **leaf**.

• Non-leaf nodes are called **internal** nodes.

• The **level number** of a node in a rooted tree is the length of the path from the root to that node.
• The level number of $x$ is 4.

• Don’t ask me why computer scientists plant their trees upside down.
Binary Trees and Beyond

- A rooted tree is called a **binary tree** if the out degree of each node is 0, 1, or 2.
- A rooted tree is called a **complete binary tree** if the out degree of each node is 0 or 2.
- A rooted tree is called an **m-ary tree** if the out degree of each node is at most $m$.
- An $m$-ary tree is called a **complete $m$-ary tree** if the out degree of each node is 0 or $m$. 
A Complete Binary Tree

root
Properties of Complete $m$-Ary Trees

**Theorem 89** Let $T$ be a complete $m$-ary tree with $n$ nodes, \( \ell \) leaves, and $i$ internal nodes. Then

1. \( n = mi + 1 \).
2. \( \ell = (m - 1)i + 1 \).
3. \( i = (\ell - 1)/(m - 1) = (n - 1)/m. \)

- Need to remove $m$ leaves to “expose” one internal node.
- Now inductively, $n - m = m(i - 1) + 1$, proving property 1.
- Observe that $\ell = n - i = mi + 1 - i = (m - 1)i + 1$.
- Property 3 merely restates properties 1 and 2.
A Numerical Example Based on p. 684

• There, \( m = 2 \), \( n = 11 \), \( \ell = 6 \), and \( i = 5 \).

• We verify the three properties of Theorem 89 below.
  \[ n = mi + 1: \quad 11 = 2 \times 5 + 1. \]
  \[ \ell = (m - 1)i + 1: \quad 6 = (2 - 1) \times 5 + 1. \]
  \[ i = (\ell - 1)/(m - 1) = (n - 1)/m: \]
  \[ 5 = (6 - 1)/(2 - 1) = (11 - 1)/2. \]

• All are satisfied.
A Corollary for Complete Binary Trees

**Corollary 90** Let $T$ be a complete binary tree with $\ell$ leaves and $i$ internal nodes. Then $i = \ell - 1 = (n - 1)/2$.

- Apply Theorem 89(3) (p. 685) with $m = 2$. 
Additional Properties of Complete Trees

**Theorem 91** Let $T$ be a complete $m$-ary tree with $n$ nodes and $\ell$ leaves. Then

1. $n = (m\ell - 1)/(m - 1)$.
2. $\ell = [(m - 1)n + 1]/m$.

- Let $i$ be the number internal nodes.
- From Theorem 89(1) (p. 685), $n = mi + 1$.
- From Theorem 89(3) (p. 685), $i = (\ell - 1)/(m - 1)$.
- Combine the two to obtain

$$n = m[(\ell - 1)/(m - 1)] + 1 = (m\ell - 1)/(m - 1).$$
Of Height and Balance

• Let $T$ be a rooted tree.

• If $h$ is the largest level number achieved by a leaf of $T$, then $T$ is said to have **height** $h$.
  
  – The tree on p. 682 has height 4.

• A rooted tree of height $h$ is said to be **balanced** if the level number of every leaf is $h - 1$ or $h$. 
Maximum Height of Binary Trees

Lemma 92  The maximum height of a rooted binary tree with \( n \) nodes is \( n - 1 \).

- A rooted binary tree achieves the maximum height when it forms a line.
- A line with \( n \) nodes has a length of \( n - 1 \).
- The result holds for rooted \( m \)-ary trees as well.\(^a\)

\(^a\)Contributed by Mr. Asger K. Pedersen (T02202107) on June 5, 2014.
Height and Number of Leaves

**Theorem 93** Consider a complete \( m \)-ary tree of height \( h \) with \( \ell \) leaves. Then \( \ell \leq m^h \) (equivalently, \( h \geq \lceil \log_m \ell \rceil \)).

- True when \( h = 1 \) as \( T \) is a tree with a root and \( \ell = m \) leaves.
- Assume the theorem holds for trees of height less than \( h \).
- Consider a tree with height \( h \) and \( \ell \) leaves.
The Proof (concluded)

- It has \( m \) subtrees \( T_1, T_2, \ldots, T_m \) at each of the children of the root.
- Let \( \ell_i \) be \( T_i \)'s number of leaves and \( h_i \leq h - 1 \) be \( T_i \)'s height.
- \( \ell_i \leq m^{h_i} \leq m^{h-1} \) by the induction hypothesis.
- So \( \ell = \ell_1 + \ell_2 + \cdots + \ell_m \leq m(m^{h-1}) = m^h \).
Height and Number of Leaves of Balanced Trees

**Corollary 94** Consider a balanced complete $m$-ary tree with $\ell$ leaves. Then its height $h$ equals $\lceil \log_m \ell \rceil$.

- $\ell \leq m^h$ by Theorem 93 (p. 691).
- $m^{h-1} < \ell$ because there are already $m^{h-1}$ nodes with a level number of $h-1$.
- Hence
  $\lceil \log_m \ell \rceil \leq h < \log_m \ell + 1 \leq \lceil \log_m \ell \rceil + 1$.
- As $h$ must be an integer, $h = \lceil \log_m \ell \rceil$. 