Number of Palindromes Revisited

- A palindrome is a composition for $m \in \mathbb{Z}^+$ that reads the same left to right as right to left (p. 107).

- Let $a_n$ denote the number of palindromes for $n$.

- Clearly, $a_1 = 1$ and $a_2 = 2$.

- Given each palindrome for $n$, we can do two things.
  - Add 1 to the first and last summands to obtain a palindrome for $n + 2$.
    * So $1 + 3 + 1$ becomes $2 + 3 + 2$.
  - Insert summand 1 to the start and end to obtain a palindrome for $n + 2$.
    * So $1 + 3 + 1$ becomes $1 + 1 + 3 + 1 + 1$. 
The Proof (continued)

• Hence $a_{n+2} = 2a_n$, $n \geq 1$.

• The characteristic equation $r^2 - 2 = 0$ has two roots $\pm \sqrt{2}$.

• The general solution is hence

$$a_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n.$$ 

• Solve\(^a\)

\[
\begin{align*}
1 &= a_1 = \sqrt{2} \left(c_1 - c_2\right), \\
2 &= a_2 = 2 \left(c_1 + c_2\right), \\
\end{align*}
\]

for $c_1 = (1 + \frac{1}{\sqrt{2}})/2$ and $c_2 = (1 - \frac{1}{\sqrt{2}})/2$.

\(^a\)This time, we are not retrofitting.
The Proof (concluded)

• The number of palindromes for $n$ therefore equals

\[
a_n = \frac{1 + \frac{1}{\sqrt{2}}}{2} (\sqrt{2})^n + \frac{1 - \frac{1}{\sqrt{2}}}{2} (-\sqrt{2})^n
\]

\[
= \begin{cases} 
\frac{1+\frac{1}{\sqrt{2}}}{2} 2^{n/2} + \frac{1-\frac{1}{\sqrt{2}}}{2} 2^{n/2}, & \text{if } n \text{ is even,} \\
\frac{1+\frac{1}{\sqrt{2}}}{2} \sqrt{2} 2^{(n-1)/2} - \frac{1-\frac{1}{\sqrt{2}}}{2} \sqrt{2} 2^{(n-1)/2}, & \text{if } n \text{ is odd,}
\end{cases}
\]

\[
= \begin{cases} 
2^{n/2}, & \text{if } n \text{ is even,} \\
2^{(n-1)/2}, & \text{if } n \text{ is odd,}
\end{cases}
\]

\[
= 2^{\lfloor n/2 \rfloor}.
\]

• This matches Theorem 21 (p. 109).
An Example: A Third-Order Relation

- Consider

\[ 2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \]

with \( a_0 = 0, \ a_1 = 1, \) and \( a_2 = 2. \)

- The characteristic equation \( 2r^3 - r^2 - 2r + 1 = 0 \) has three distinct real roots: 1, -1, and 0.5.

- The general solution is

\[
a_n = c_1 1^n + c_2 (-1)^n + c_3 (1/2)^n
\]

\[
= c_1 + c_2 (-1)^n + c_3 (1/2)^n.
\]
An Example: A Third-Order Relation (concluded)

- Solving the three initial conditions, we have\(^a\)

\[
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 0.5 \\
1^2 & (-1)^2 & 0.5^2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}.
\]

- The solutions are \(c_1 = 2.5\), \(c_2 = 1/6\), and \(c_3 = -8/3\).

\(^a\)Or see Eq. (75) on p. 516.
The Case of Complex Roots

- Consider
  \[ a_n = 2(a_{n-1} - a_{n-2}) \]
  with \( a_0 = 1 \) and \( a_1 = 2 \).

- The characteristic equation \( r^2 - 2r + 2 = 0 \) has two distinct complex roots \( 1 \pm i \).

- The general solution is
  \[ a_n = c_1(1 + i)^n + c_2(1 - i)^n. \]
The Case of Complex Roots (concluded)

• Solve the two initial conditions for $c_1 = (1 - i)/2$ and $c_2 = (1 + i)/2$.

• The particular solution becomes\(^{a}\)

\[
a_n = (1 + i)^{n-1} + (1 - i)^{n-1}
\]
\[
= (\sqrt{2})^n [\cos(n\pi/4) + \sin(n\pi/4)].
\]

---

\(^{a}\)An equivalent one is $a_n = (\sqrt{2})^{n+1} \cos((n - 1)\pi/4)$ by Mr. Tunglin Wu (B00902040) on May 17, 2012.
$k$th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Repeated Real Roots

• Consider the recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0,$$

where $C_n, C_{n-1}, \ldots$ are real constants, $C_n \neq 0$, $C_{n-k} \neq 0$.

• Let $r$ be a characteristic root of multiplicity $m$, where $2 \leq m \leq k$, of the characteristic equation

$$f(x) = C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0.$$

• The general solution that involves $r$ has the form

$$(A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1}) r^n,$$  \hspace{1cm} (81)

with $A_0, A_1, \ldots, A_{m-1}$ are constants to be determined.
The Proof

• If \( f(x) \) has a root \( r \) of multiplicity \( m \), then
  \[
  f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0.
  \]

• Because \( r \neq 0 \) is a root of multiplicity \( m \), it is easy to check that

\[
0 = r^{n-k} f(r),
\]
\[
0 = r (r^{n-k} f(r))',
\]
\[
0 = r (r (r^{n-k} f(r))')',
\]
\[
\vdots
\]
\[
0 = \overbrace{r (\cdots r (r^{n-k} f(r))')'}^{m-1} \cdots \overbrace{r^{m-1}}^{m-1}'.
\]
The Proof (continued)

• Note that we differentiate and then multiply by \( r \) before iterating.

• These give

\[
0 = C_n r^n + C_{n-1} r^{n-1} + \cdots + C_{n-k} r^{n-k},
\]
\[
0 = C_n n r^n + C_{n-1} (n-1) r^{n-1} + \cdots + C_{n-k} (n-k) r^{n-k},
\]
\[
0 = C_n n^2 r^n + C_{n-1} (n-1)^2 r^{n-1} + \cdots + C_{n-k} (n-k)^2 r^{n-k},
\]
\[
\vdots
\]
The Proof (continued)

- Now, \( a_n = n^k r^n \), \( 0 \leq k \leq m - 1 \), is indeed a solution because the \( k \)th row above says

\[
0 = C_n n^k r^n + C_{n-1} (n-1)^k r^{n-1} + \cdots + C_{n-k} (n-k)^k r^{n-k}
\]

\[
= C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k}.
\]
The Proof (continued)

- From Eq. (73) on p. 511, $r^n, nr^n, n^2r^n, \ldots, n^{m-1}r^n$ form a fundamental set if

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 \\
r & r & \cdots & r \\
r^2 & 2r^2 & \cdots & 2^{m-1}r^2 \\
\vdots & \vdots & \ddots & \vdots \\
r^{m-1} & (m-1)r^{m-1} & \cdots & (m-1)^{m-1}r^{m-1}
\end{vmatrix} \neq 0.
\]

- But it is a Vandermonde matrix in disguise.

\(^a\)The \(i\)th row sets \(n = i - 1, i = 1, 2, \ldots, m.\)
The Proof (concluded)

• In fact, after deleting the first row and column, the determinant equals

\[(m - 1)! r^{1+2+\cdots+(m-1)} \]

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2^{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (m - 1) & \cdots & (m - 1)^{m-2}
\end{vmatrix} \neq 0.
\]
Nonhomogeneous Recurrence Relations

• Consider

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \]  \hspace{1cm} (82)

• If \( a_n = a_{n-1} + f(n) \), then the solution is

\[ a_n = a_0 + \sum_{i=1}^{n} f(i). \]

  – A closed-form formula exists if one for \( \sum_{i=1}^{n} f(i) \)
    does.
Nonhomogeneous Recurrence Relations (concluded)

• In general, no failure-free methods exist except for specific $f(n)$s.
  – See pp. 441–2 of the textbook (4th ed.).
  – See p. 532 of Rosen (2012) when $f(n)$ is the product of a polynomial in $n$ and the $n$th power of a constant.
**Examples** \((c, c_1, c_2, \ldots \text{ Are Arbitrary Constants})\)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{n+1} - a_n = 0)</td>
<td>(a_n = c)</td>
</tr>
<tr>
<td>(a_{n+1} - a_n = 1)</td>
<td>(a_n = n + c)</td>
</tr>
<tr>
<td>(a_{n+1} - a_n = n)</td>
<td>(a_n = n(n - 1)/2 + c)</td>
</tr>
<tr>
<td>(a_{n+2} - 3a_{n+1} + 2a_n = 0)</td>
<td>(a_n = c_1 + c_22^n)</td>
</tr>
<tr>
<td>(a_{n+2} - 3a_{n+1} + 2a_n = 1)</td>
<td>(a_n = c_1 + c_22^n - n)</td>
</tr>
<tr>
<td>(a_{n+2} - a_n = 0)</td>
<td>(a_n = c_1 + c_2(-1)^n)</td>
</tr>
<tr>
<td>(a_{n+1} = a_n/(1 + a_n))</td>
<td>(a_n = c/(1 + cn))</td>
</tr>
</tbody>
</table>
Trial and Error

- Consider $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- Calculations show that $a_2 = 4$ and $a_3 = 12$.
- Conjecture:
  \[ a_n = n2^{n-1}. \] (83)
- Verify that, indeed,
  \[ (n + 1)2^n = 2(n2^{n-1}) + 2^n, \]
  and $a_1 = 1$. 
Application: Number of Edges of a Hasse Diagram

- Let $a_n$ be the number of edges of the Hasse diagram for the partial order $(2\{1,2,\ldots,n\}, \subseteq)$.

- Consider the Hasse diagrams $H_1$ for $(2\{1,2,\ldots,n\}, \subseteq)$ and $H_2$ for $\left(\{T \cup \{n+1\} : T \subseteq \{1, 2, \ldots, n\}\}, \subseteq\right)$.
  - $H_1$ and $H_2$ are “isomorphic.”

- The Hasse diagram for $(2\{1,2,\ldots,n+1\}, \subseteq)$ is constructed by adding an edge from each node $T$ of $H_1$ to node $T \cup \{n+1\}$ of $H_2$.

- Hence $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.

- The desired number has been solved in Eq. (83) on p. 551.
Illustration with \( (2^{\{1,2,3\}}, \subseteq) \)
Trial and Error Again

- Consider \( a_{n+1} - Aa_n = B \).

- Calculations show that

\[
\begin{align*}
a_1 &= Aa_0 + B, \\
a_2 &= Aa_1 + B = A^2a_0 + B(A + 1), \\
a_3 &= Aa_2 + B = A^3a_0 + B(A^2 + A + 1).
\end{align*}
\]

- Conjecture (easily verified by substitution):

\[
a_n = \begin{cases} 
A^n a_0 + B \frac{A^{n-1}}{A-1}, & \text{if } A \neq 1 \\
a_0 + Bn, & \text{if } A = 1.
\end{cases}
\] (84)
Financial Application: Compound Interest

- Consider \( a_{n+1} = (1 + r) a_n \).
  - Deposit grows at a period interest rate of \( r > 0 \).
  - The initial deposit is \( a_0 \) dollars.

- The solution is obviously
  \[
  a_n = (1 + r)^n a_0.
  \]

- The deposit therefore grows exponentially with time.

---

\(^a\)“In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect,” wrote Joseph Alois Schumpeter (1883–1950) in *Capitalism, Socialism and Democracy* (1942).
Financial Application: Amortization

• Consider \( a_{n+1} = (1 + r) a_n - M. \)
  
  – The initial loan amount is \( a_0 \) dollars.
  
  – The monthly payment is \( M \) dollars.
  
  – The outstanding loan principal after the \( n \)th payment is \( a_n \).

• By Eq. (84) on p. 554, the solution is

\[
a_n = (1 + r)^n a_0 - M \frac{(1 + r)^n - 1}{r}.
\]
The Proof (concluded)

• What is the unique monthly payment $M$ for the loan to be closed after $k$ monthly payments?

• Set $a_k = 0$ to obtain

$$a_k = (1 + r)^k a_0 - M \frac{(1 + r)^k - 1}{r} = 0.$$ 

• Hence

$$M = \frac{(1 + r)^k a_0 r}{(1 + r)^k - 1}.$$ 

• This is standard calculation for home mortgages and annuities.\(^a\)

\(^a\)Lyuu (2002).
Trial and Error a Third Time

• Consider the more general \( a_{n+1} - Aa_n = BC^n \).

• Calculations show that

\[
\begin{align*}
a_1 &= Aa_0 + B, \\
a_2 &= Aa_1 + BC = A^2a_0 + B(A + C), \\
a_3 &= Aa_2 + BC^2 = A^3a_0 + B(A^2 + AC + C^2).
\end{align*}
\]

• Conjecture (easily verified by substitution):

\[
a_n = \begin{cases} 
A^n a_0 + B \frac{A^n - C^n}{A - C} & \text{if } A \neq C \\
A^n a_0 + BA^{n-1}n & \text{if } A = C
\end{cases}.
\]

(85)
Application: Runs of Binary Strings

• A run is a maximal consecutive list of identical objects (p. 111).
  – Binary string “0 0 1 1 1 0” has 3 runs.

• Let $r_n$ denote the total number of runs determined by the $2^n$ binary strings of length $n$.

• First, $r_1 = 2$.
  – Each of “0” and “1” has 1 run.

• In general, suppose we append a bit to every $(n - 1)$-bit string $b_1b_2\cdots b_{n-1}$ to make $b_1b_2\cdots b_{n-1}b_n$. 
The Proof (continued)

• For those with $b_{n-1} = b_n$ (i.e., the last 2 bits are identical), the total number of runs does not change.
  – The total number of runs remains $r_{n-1}$.

• For those with $b_{n-1} \neq b_n$ (i.e., the last 2 bits are distinct), the total number of runs increases by 1 for each $(n - 1)$-bit string.
  – There are $2^{n-1}$ of them.
  – So the total number of runs becomes $r_{n-1} + 2^{n-1}$.

• Hence

$$r_n = 2r_{n-1} + 2^{n-1}, n \geq 2.$$
The Proof (concluded)

• By Eq. (85) on p. 558,

\[ r_n = 2^n r_0 + 2^{n-1} n. \]

• To make sure that \( r_1 = 2 \), it is easy to see that \( r_0 = 1/2 \).

• Hence

\[ r_n = 2^{n-1} + 2^{n-1} n = 2^{n-1}(n + 1). \]

  – The recurrence is identical to that for the number of edges of a Hasse diagram (p. 552) except for the initial condition.

  – Its slightly different solution appeared in Eq. (83) on p. 551, \( a_n = n2^{n-1} \).
Method of Undetermined Coefficients

• Recall Eq. (82) on p. 548, repeated below:

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \]  \hspace{1cm} (86)

• Let \( a_n^{(h)} \) denote the general solution of the associated homogeneous relation (with \( f(n) = 0 \)).

• Let \( a_n^{(p)} \) denote a particular solution of the nonhomogeneous relation.

• Then

\[ a_n = a_n^{(h)} + a_n^{(p)}. \]

• All the entries in the table on p. 550 fit the claim.
Conditions for the General Solution

Similar to Theorem 69 (p. 511), we have the following theorem.

**Theorem 70** Let \( a_n^{(p)} \) be any particular solution of the nonhomogeneous recurrence relation Eq. (86) on p. 562. Let

\[
a_n^{(h)} = C_1 a_n^{(1)} + C_2 a_n^{(2)} + \cdots + C_k a_n^{(k)}
\]

be the general solution of its homogeneous version as specified in Theorem 69. Then \( a_n^{(h)} + a_n^{(p)} \) is the general solution of Eq. (86) on p. 562.
Solution Techniques

- Typically, one finds the general solution of its homogeneous version $a_n^{(h)}$ first.
- Then one finds a particular solution $a_n^{(p)}$ of the nonhomogeneous recurrence relation Eq. (86) on p. 562.
- Make sure $a_n^{(p)}$ is “independent” of $a_n^{(h)}$.
- Finally, use the initial conditions to nail down the coefficients of $a_n^{(h)}$.
- Output $a_n^{(h)} + a_n^{(p)}$. 
\[ a_{n+1} - Aa_n = B \ 	ext{Revisited} \]

- Recall that the general solution is \( a_n^{(h)} = cA^n \).
- A particular solution is
  \[
  a_n^{(p)} = \begin{cases} 
  B/(1-A) & \text{if } A \neq 1 \\
  Bn & \text{if } A = 1
  \end{cases}.
  \]
- So \( a_n = cA^n + a_n^{(p)} \).
- In particular,
  \[
  c = a_0 - a_0^{(p)} = \begin{cases} 
  a_0 - B/(1-A) & \text{if } A \neq 1 \\
  a_0 & \text{if } A = 1
  \end{cases}.
  \]
\( a_{n+1} - Aa_n = B \) Revisited (concluded)

- The solution matches Eq. (84) on p. 554.
- We can rewrite the solution as

\[
a_n = \begin{cases} 
  A^n [a_0 - a_n^{(p)}] + a_n^{(p)}, & \text{if } A \neq 1 \\
  a_0 + a_n^{(p)}, & \text{if } A = 1
\end{cases} \quad (87)
\]
Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 7^n$ with $a_0 = 2$

- $a_n^{(h)} = c \times 3^n$, because the characteristic equation has the nonzero root 3.

- We propose $a_n^{(p)} = a \times 7^n$.

- Place $a \times 7^n$ into the relation to obtain
  $$a \times 7^n - 3a \times 7^{n-1} = 5 \times 7^n.$$

- Hence $a = 35/4$ and $a_n^{(p)} = (35/4) \times 7^n = (5/4) \times 7^{n+1}$.

- The general solution is $a_n = c \times 3^n + (5/4) \times 7^{n+1}$.

- Now, $c = -27/4$ because $a_0 = 2 = c + (5/4) \times 7$.

- So the solution is $a_n = -(27/4) \times 3^n + (5/4) \times 7^{n+1}$. 
Nonhomogeneous \( a_n - 3a_{n-1} = 5 \times 3^n \) with \( a_0 = 2 \)

- As before, \( a_n^{(h)} = c \times 3^n \).
- But this time \( a_n^{(h)} \) and \( f(n) = 5 \times 3^n \) are not “independent.”
- So propose \( a_n^{(p)} = an \times 3^n \).
- Plug \( an \times 3^n \) into the relation to obtain
\[
an \times 3^n - 3a(n - 1) \times 3^{n-1} = 5 \times 3^n.
\]
- Hence \( a = 5 \) and \( a_n^{(p)} = 5n \times 3^n \).
- The general solution is \( a_n = c \times 3^n + 5n \times 3^n \).
- Finally we find that \( c = 2 \) with use of \( a_0 = 2 \).
Nonhomogeneous $a_{n+1} - 2a_n = n + 1$ with $a_0 = 4$

- From Eq. (84) on p. 554, $a_n^{(h)} = c \times 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute this particular solution into the relation to yield
  \[ a(n + 1) + b - 2(an + b) = n + 1. \]
- Rearrange the above to obtain
  \[ (-a - 1)n + (a - b - 1) = 0. \]
- This holds for all $n$ if $a = -1$ and $b = -2$.  

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The Proof (concluded)

• Hence $a_n^{(p)} = -n - 2$.

• The general solution is

$$a_n = c \times 2^n - n - 2.$$  

• Use the initial condition

$$4 = a_0 = c - 2$$

to obtain $c = 6$.

• The solution to the complete relation is

$$a_n = 6 \times 2^n - n - 2.$$
Nonhomogeneous $a_{n+1} - a_n = 2n + 3$ with $a_0 = 1$

- This equation is very similar to the previous one: $a_{n+1} - 2a_n = n + 1$.

- First, $a_n^{(h)} = d \times 1^n = d$.

- If one guesses $a_n^{(p)} = an + b$ as before, then

  $$a_{n+1} - a_n = a(n + 1) + b - an - b = a,$$

  which cannot be right.\(^a\)

- So we guess $a_n^{(p)} = an^2 + bn + c$.

\(^a\)Contributed by Mr. Yen-Chieh Sung (B01902011) on June 17, 2013.
The Proof (continued)

- Substitute this particular solution into the relation to yield

\[ a(n + 1)^2 + b(n + 1) + c - (an^2 + bn + c) = 2n + 3. \]

- Simplify the above to obtain

\[ 2an + (a + b) = 2n + 3. \]

- Hence \( a = 1 \) and \( b = 2. \)
- Hence \( a_n^{(p)} = n^2 + 2n + c. \)
- The general solution is \( a_n = n^2 + 2n + c. \)

\(^{\text{aWe merge } d \text{ into } c.}\)
The Proof (concluded)

- Use the initial condition

\[ 1 = a_0 = c \]

to obtain \( c = 1 \).

- The solution to the complete relation is

\[ a_n = n^2 + 2n + 1 = (n + 1)^2. \]

- It is very different from the solution to the previous example: \( a_n = 6 \times 2^n - n - 2 \).
Nonhomogeneous $a_{n+2} - 3a_{n+1} + 2a_n = 2$ with $a_0 = 0$ and $a_1 = 2$

- The characteristic equation $r^2 - 3r + 2 = 0$ has roots 2 and 1.
- So $a_n^{(h)} = c_1 1^n + c_2 2^n = c_1 + c_2 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute $a_n^{(p)}$ into the relation to yield

$$a(n + 2) + b - 3[a(n + 1) + b] + 2(an + b) = 2.$$  

- Rearrange the above to obtain $a = -2$.
- Hence $a_n^{(p)} = -2n + b$. 
The Proof (concluded)

• The general solution is now \( a_n = c_1 + c_2 2^n - 2n \).\(^a\)

• Use the initial conditions

\[
0 = a_0 = c_1 + c_2, \\
2 = a_1 = c_1 + 2c_2 - 2.
\]

to obtain \( c_1 = -4 \) and \( c_2 = 4 \).

• The solution to the complete relation is

\[
a_n = -4 + 2^{n+2} - 2n.
\]

\(^a\)We merge \( b \) into \( c_1 \).
The Method of Generating Functions (Recall p. 524)

• Consider the relation \(a_n - 3a_{n-1} = n\) with \(a_0 = 1\).

• Let \(f(x) = \sum_{n=0}^{\infty} a_n x^n\) be the generating function for \(a_0, a_1, \ldots\).

• From the recurrence equation,

\[
\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n = \sum_{n=1}^{\infty} nx^n.
\]

• \(f(x) - a_0 - 3x f(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}\) from p. 441.

• Hence

\[
f(x) = \frac{x}{(1-x)^2} + 1.
\]
The Method of Generating Functions (continued)

- Now,

\[
\begin{align*}
f(x) &= \frac{1}{1-3x} + \frac{x}{(1-x)^2(1-3x)} \\
&= \frac{7/4}{1-3x} + \frac{-1/4}{1-x} + \frac{-1/2}{(1-x)^2}
\end{align*}
\]

by a partial fraction decomposition.
The Method of Generating Functions (continued)

- \( \frac{7/4}{1-3x} = (7/4) \frac{1}{1-3x} = (7/4) \sum_{n=0}^{\infty} (3x)^n \).

- \( \frac{-1/4}{1-x} = -(1/4) \frac{1}{1-x} = -(1/4) \sum_{n=0}^{\infty} x^n \).

- \( \frac{-1/2}{(1-x)^2} = -(1/2) \frac{1}{(1-x)^2} = -(1/2) \sum_{n=0}^{\infty} (n+1) x^n \) from p. 440.
The Method of Generating Functions (concluded)

- Now,

\[ f(x) = \left( \frac{7}{4} \right) \sum_{n=0}^{\infty} 3^n x^n - \left( \frac{1}{4} \right) \sum_{n=0}^{\infty} x^n - \left( \frac{1}{2} \right) \sum_{n=0}^{\infty} (n + 1) x^n. \]

- So

\[ a_n = \left( \frac{7}{4} \right) 3^n - \left( \frac{1}{4} \right) - \left( \frac{1}{2} \right)(n + 1). \]

- The methodology should be clear.
The Method of Generating Functions for
\( a_{n+1} - a_n = 3^n \) with \( a_0 = 1 \)

- Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be the generating function for \( a_0, a_1, \ldots \).

- From the recurrence equation,
  \[
  \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} 3^n x^{n+1}.
  \]

- \( f(x) - a_0 - xf(x) = x \sum_{n=0}^{\infty} (3x)^n = \frac{x}{1-3x} \).

- This implies that
  \[
  f(x) = \frac{x}{1-3x} + 1 = \frac{1/2}{1-3x} + \frac{1/2}{1-x} = \left( \frac{1}{2} \right) \sum_{n=0}^{\infty} (3^n + 1) x^n.
  \]

- Hence \( a_n = (3^n + 1)/2 \).
The Method of Generating Functions for $a_{n+1} - Aa_n = B$ Again

- Assume $A \neq 1$.
- We want to obtain Eq. (87) on p. 566 by the method of generating functions.
- Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots$. 
The Proof (continued)

- Then $\sum_{n=0}^{\infty} a_{n+1}x^n - \sum_{n=0}^{\infty} Aa_nx^n = \sum_{n=0}^{\infty} Bx^n$.
- So 
  \[
  \frac{f(x) - a_0}{x} - Af(x) = \frac{B}{1 - x}
  \]

from p. 437.
The Proof (continued)

- Simplify the identity to yield

\[
\begin{align*}
  f(x) &= \frac{a_0}{1 - Ax} + \frac{Bx}{(1 - x)(1 - Ax)} \\
  &= \frac{a_0}{1 - Ax} + \frac{B}{1 - A} \left( \frac{1}{1 - x} - \frac{1}{1 - Ax} \right) \\
  &= \frac{a_0}{1 - Ax} + a_n^{(p)} \left( \frac{1}{1 - x} - \frac{1}{1 - Ax} \right) \\
  &= \left[ a_0 - a_n^{(p)} \right] \frac{1}{1 - Ax} + a_n^{(p)} \frac{1}{1 - x},
\end{align*}
\]

where \( a_n^{(p)} \equiv B/(1 - A) \).
The Proof (concluded)

• From p. 437,

\[ f(x) = \left[ a_0 - a_n^{(p)} \right] \sum_{n=0}^{\infty} A^n x^n + a_n^{(p)} \sum_{n=0}^{\infty} x^n. \]

− Note that \( a_n^{(p)} \) is independent of \( n \).

• So

\[ a_n = A^n \left[ a_0 - a_n^{(p)} \right] + a_n^{(p)}, \]

matching the earlier solution on p. 566 as desired.
Convolutions

• Consider the following recurrence equation,

\[ b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0. \]

• Then

\[
\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0) x^{n+1}.
\]

• Let \( f(x) = \sum_{n=0}^{\infty} b_n x^n. \)

• Then \( f(x) - b_0 = x f^2(x) \) from p. 446.
The Proof (continued)

• When \( b_0 = 1 \),

\[
f(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
\]

• Pick

\[
f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}
\]

to match \( b_0 \).\(^a\)

• By Eq. (62) on p. 458,

\[
\sqrt{1 - 4x} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n x^n.
\]

\(^a f(0) = \infty \) if one picks + (Graham, Knuth, Patashnik (1989)).
The Proof (concluded)

- Now,

\[
\binom{1/2}{n}(-4)^n = \frac{1}{n!} \left( \frac{1}{2} - 1 \right) \cdots \left( \frac{1}{2} - n + 1 \right) (-4)^n = -\frac{1}{2n-1} \binom{2n}{n}.
\]

- So

\[
f(x) = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2(2n-1)} x^{n-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^{n-1}
\]

\[
= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^n,
\]

the Catalan numbers (recall Eq. (17) on p. 118)!
An Example

- It is easy to verify that
  \[ f(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots. \]

- The coefficients are indeed
  \[
  \frac{0}{1}, \frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \frac{10}{6}, \cdots.
  \]
A Binary Tree$^a$

$^a$Gustav Kirchhoff (1824–1887).
Number of Rooted Binary Trees

- There is a distinct node called the root.
- A rooted binary tree is ordered if the left and right branches are considered distinct.
- What is the number $b_n$ of rooted ordered binary trees on $n$ nodes?
Illustration: $b_3 = 5$
Number of Rooted Binary Trees: The Formula

- \( b_0 = 1 \), as it is the empty tree.
- Recursively,
  \[
  b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0.
  \]
  - \( b_i b_{n-i} \): \( i \) nodes on the left and \( n - i \) nodes on the right, \( 0 \leq i \leq n \).
- So \( b_n \) is the \( n \)th Catalan number by Eq. (88) on p. 587:
  \[
  b_n = \frac{\binom{2n}{n}}{n+1}.
  \]
An Introduction to Graph Theory
If 50 million people believe a foolish thing,
it’s still a foolish thing.
— George Bernard Shaw (1856–1950)
Graphs

- Let $V$ be a finite nonempty set of nodes.
- Let $E \subseteq V \times V$ be a set of edges.
- $G = (V, E)$ is the directed graph (or digraph) made up of the node set $V$ and the edge set $E$.
- When $E$ is considered to consist of unordered pairs, $(V, E)$ is called an undirected graph.
- A graph is loop-free if it contains no loops.
- A multigraph allows multiple edges between nodes.

\(^a\)Founded by Leonhard Euler in 1736.
\(^b\)Assumed unless stated otherwise.
Graphs (concluded)

- For an undirected graph, we typically use \( \{x, y\} \) to represent an edge.
- For a digraph, we always use \( (x, y) \) to represent an edge.
Illustration of Graphs

• In the following graph $G$,

\[ V = \{a, b, c, d, e, f, g, h\} \]
\[ E = \{\{a, b\}, \{a, e\}, \{a, f\}, \{b, c\}, \{b, g\}, \{b, f\}, \]
\[ \{f, g\}, \{f, h\}, \{c, d\}, \{c, h\}, \{c, g\}, \]
\[ \{d, e\}, \{d, h\}, \{g, h\}, \{h, e\}\}. \]
Applications of Graph Theory

- Representation of networks, both structured ones like interconnection networks and unstructured ones like the telephone network or the social network.
- Natural representation of relations (p. 333).
- Practically any computation can be described as a graph.
- Optimization problems such as circuit layout.
- Physical systems such as ferromagnetism.
- Social networks.
- ...
Additional Notions

• Let \( G = (V, E) \) be a graph (directed or otherwise).

• \( G_1 = (V_1, E_1) \) is called a subgraph of \( G \) if
  - \( \emptyset \neq V_1 \subseteq V \).
  - \( E_1 \subseteq V_1 \times V_1 \).
  - \( E_1 \subseteq E \).

• \( G_1 \) is an induced subgraph of \( G \) if it is a subgraph of \( G \) and \( E_1 = E \cap (V_1 \times V_1) \).

• An undirected graph \( G \) is connected if there is a path between any two distinct nodes of \( G \).

• A component is a maximal subgraph that is connected.
Illustration of Subgraphs
All Kinds of Walks on Undirected Graphs

• A walk from $x$ to $y$ is a finite sequence of non-loop edges connecting $x$ and $y$.

• The length of a walk is the number of edges in it.

• A walk from $x$ to $y$ where $x \neq y$ is called an open walk.

• A walk from $x$ to itself is called a closed walk.

• A walk without repeated edges is called a trail.

• A closed trail is called a circuit.
All Kinds of Walks on Undirected Graphs (concluded)

- A walk without repeated nodes is a (simple) path.
- A closed path is called a cycle.
  - A cycle must be a circuit, but not vice versa.
- By convention, a cycle has at least 3 distinct edges.
- A cycle of even length is called an even cycle; a cycle of odd length is called an odd cycle.
- These definitions apply to digraphs with minimal changes.
- A digraph that has no cycles is called acyclic.
Illustration of Walks

- $(b, c, g, b, f)$ is a trail of length 4.
- $(a, b, c)$ is a path of length 2.
- $(a, b, c, d, e, a)$ is a cycle of length 5.
- $(g, b, c, g, h, e, a, f, g)$ is a circuit but not a cycle.
Partial Order and Its Digraph Representation

- The digraph representation of a partial order (p. 340) must be acyclic.
  - Recall p. 345.

- Any acyclic digraph entails a partial order.
  - Take the transitive closure of the digraph.
  - The resulting digraph clearly remains acyclic.
  - Add a loop to every node.
  - It is not hard to check that the digraph’s associated relation satisfies the definition of partial order.
Transitive Closure of a Digraph
Diameter

- Let $G(V, E)$ be an undirected graph.
- The **distance** between nodes $x, y \in V$ (or $d(x, y)$) is the minimum length of all the paths between $x$ and $y$.
- The **diameter** $d(G)$ of $G$ is the maximum distance over all pairs of nodes of $G$.
  - So any two nodes must have distance at most $d(G)$ between them.
- Diameter can be computed by an efficient all-pair-shortest-paths algorithm.$^a$

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$^a$Floyd (1962); Roy (1959); Warshall (1962).
Complete Graphs

• Let $V$ be a set of $n$ nodes.

• The **complete graph** on $V$, denoted $K_n$, is a loop-free undirected graph.
  - There is an edge between any pair of distinct nodes.
  - $K_n$ has $\binom{n}{2}$ edges.
  - Depending on applications, sometimes (self-)loops are allowed.

• The diameter of $K_n$ is clearly one.
$K_{17}$
Complete Graphs (concluded)

- There are \( \binom{n}{i} \) ways to pick \( i \) nodes from \( K_n \).
- As there are \( \binom{i}{2} \) pairs of nodes, there are \( 2^{\binom{i}{2}} \) ways to pick the edges.
- Hence \( K_n \) has
  \[
  \sum_{i=1}^{n} \binom{n}{i} 2^{\binom{i}{2}}
  \]
  subgraphs.
- Can you simplify it?
An Inequality Relating $|V|$ and $|E|$ 

Lemma 71 Let $G = (V, E)$ be an undirected graph. Then 

$$|V| \geq \frac{1 + \sqrt{1 + 8 \times |E|}}{2}.$$ 

- $G$ has at most $\left(\frac{|V|}{2}\right)$ edges (the complete graph).
- So $V$ must be big enough such that $\left(\frac{|V|}{2}\right) \geq |E|$.
- This results in $|V|^2 - |V| \geq 2 \times |E|$, or 

$$\left(|V| - \frac{1}{2}\right)^2 \geq \frac{1}{4} + 2 \times |E| \geq \frac{1 + 8 \times |E|}{4}.$$
Complements

- The **complement** of graph $G$, denoted $\overline{G}$, is the subgraph of $K_n$ consisting of the nodes in $G$ and all edges that are not in $G$.
  - $\overline{K_n}$, consisting of $n$ nodes and no edges is called a null graph.
A Useful Identity

- Let $G = (V, E)$ be an undirected graph.
- For each node $v \in G$, the degree of $v$, written $\text{deg}(v)$, is the number of edges in $G$ that are incident with $v$.
  - A loop is considered as two incident edges.
- Now, the handshaking theorem says
  \[
  \sum_{v \in V} \text{deg}(v) = 2 \times |E|.
  \]
  - An edge is counted twice, once at each end.

Corollary 72 For finite graphs, the number of nodes of odd degree must be even.