The Binomial Theorem$^a$

**Theorem 12**

\[(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.\]

- \((x + y)^n = (x + y)(x + y) \cdots (x + y)\).

- Each term must have the form \(x^i y^{n-i}\).

- There are \(\binom{n}{i}\) ways to pick \(i\) \(x\)'s and \(n-i\) \(y\)'s.

---

Corollaries of the Binomial Theorem

\[ 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}. \quad (6) \]

- Set \( x = y = 1 \) in the binomial theorem.

**Corollary 13** \( \binom{n}{k} \leq 2^n \) with \( 0 \leq k \leq n \).

- A part cannot be greater than the sum.
Corollaries of the Binomial Theorem (continued)

Corollary 14 \( \binom{n}{\lfloor n/2 \rfloor} \geq 2^n / n \) for \( n \geq 2 \).\(^a\)

- Note that

\[
2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2 + \binom{n}{1} + \cdots + \binom{n}{n-1}.
\]

- Now \( \binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{k} \geq 2 \) with \( 0 < k < n \) (p. 26).

- Hence

\[
2^n \geq n \binom{n}{\lfloor n/2 \rfloor}.
\]

\(^a\)Corrected by Mr. Connor J. Shinn (T03203102) on March 5, 2015.
Corollaries of the Binomial Theorem (continued)

For odd $n$,

\[
2^{n-1} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}}
\]

\[
= \binom{n}{\frac{n+1}{2}} + \binom{n}{\frac{n+3}{2}} + \cdots + \binom{n}{n}. \quad (7)
\]

- Because $\binom{n}{r} = \binom{n}{n-r}$. 

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Corollaries of the Binomial Theorem (continued)

- Set $x = 1$ and $y = -1$ in the binomial theorem to obtain

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0. \quad (8)$$

- As a by-product, when $n > 0$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

$$= \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

$$= 2^{n-1}. \quad (9)$$
Corollaries of the Binomial Theorem (continued)

\[ \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2^{2n} \quad (10) \]

because

\[
2^{2n+1} = \sum_{i=0}^{2n+1} \binom{2n+1}{i} = \sum_{i=0}^{n} \binom{2n+1}{i} + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \\
= \sum_{i=0}^{n} \binom{2n+1}{2n+1-i} + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \\
= \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} \\
= 2 \sum_{i=n+1}^{2n+1} \binom{2n+1}{i}.
\]
Corollaries of the Binomial Theorem (continued)

- In fact, it is just Eq. (7) on p. 56!
Corollaries of the Binomial Theorem (continued)

\[
\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2.
\]  \hfill (11)

- Consider

\[
f(x) = (1 + x)^n (1 + x^{-1})^n
\]

\[
= \left( \underbrace{(1 + x) \cdots (1 + x)}_{n} \right) \left( \underbrace{(1 + x^{-1}) \cdots (1 + x^{-1})}_{n} \right)
\]

\[
= \sum_{i=-n}^{n} f_i x^i.
\]

- Concentrate on the constant term \( f_0 \) of \( f(x) \).
The Proof (concluded)

• \((\frac{n}{i})^2\) is the number of ways to pick \(i\) \(x\)'s and \(i\) \(x^{-1}\)'s.

• So

\[
f_0 = \sum_{i=0}^{n} \left(\frac{n}{i}\right)^2.
\]

• Rewrite \(f(x)\) as

\[
f(x) = (1 + x)^n (1 + x)^n x^{-n} = x^{-n} (1 + x)^{2n}.
\]

• The constant term in \(f(x)\) is the coefficient of \(x^n\) in \((1 + x)^{2n}\).

• So \(f_0 = \binom{2n}{n}\).\(^a\)

\(^a\)See Lemma 28 (p. 146) for an upper bound on \(\binom{2n}{n}\).
An Alternative Proof for Eq. (11) on p. 60

- Consider a $2n$-step binomial random walk that ends at the origin.
- There are $\binom{2n}{n}$ such walks by Eq. (4) on p. 42.
- Among them, consider walks that go through position $i$ at step $n$, where $n + i$ is even.
- There are $\left(\binom{n}{(n+i)/2}\right)^2$ such walks by Eq. (4) on p. 42.
- So

$$\binom{2n}{n} = \sum_{i=-n,-n+2,\ldots,n} \left(\binom{n}{(n+i)/2}\right)^2 = \sum_{i=0}^{n} \left(\binom{n}{i}\right)^2.$$
A Combinatorial Proof\(^a\) for Eq. (11) on p. 60

- There are \(\binom{2n}{n}\) ways to pick \(n\) objects out of \(2n\) distinct objects.

- Now, divide the \(2n\) objects into two groups equally.

- There are \(\binom{n}{i}\binom{n}{n-i} = \left(\binom{n}{i}\right)^2\) ways to pick \(i\) objects from the first group and the remaining \(n-i\) objects from the second.

- As \(i\) can vary from 0 to \(n\),

\[
\binom{2n}{n} = \sum_{i=0}^{n} \left(\binom{n}{i}\right)^2.
\]

\(^a\)Contributed by Mr. Kung-Ching Lin (B00703082) on February 27, 2012.
A Fourth Proof for Eq. (11) on p. 60

- Recall Vandermonde’s convolution (p. 47):

\[
\binom{m}{n} = \sum_{i=0}^{k} \binom{k}{i} \binom{m-k}{n-i}.
\]

- Now choose \( m = 2n \) and \( k = n \) to obtain

\[
\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{2n-n}{n-i} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i}^2.
\]
Corollaries of the Binomial Theorem (continued)

\[
\binom{2n}{n} = \sum_{i=0}^{2n} (-1)^{n+i} \binom{2n}{i}^2.
\]

- Consider

\[
g(x) = (1 + x)^{2n}(1 - x^{-1})^{2n}
\]

\[
= \left( (1 + x) \cdots (1 + x) \right) \left( (1 - x^{-1}) \cdots (1 - x^{-1}) \right)
\]

\[
= \sum_{i=-2n}^{2n} g_i x^i.
\]
The Proof (concluded)

- Concentrate on the constant term $g_0$ of $g(x)$.
- $\binom{2n}{i}^2$ is the number of ways to pick $i$ “$x$” and $i$ “$-x^{-1}$”.
- So $g_0 = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i}^2$.
- Note that $(1 + x)(1 - x^{-1}) = x - x^{-1}$.
- Hence
  
  $$g(x) = (x - x^{-1})^{2n} = x^{-2n}(x^2 - 1)^{2n}.$$  

- The constant term in $g(x)$ is the coefficient of $x^{2n}$ in $(x^2 - 1)^{2n}$.
- So $g_0 = (-1)^n \binom{2n}{n}$. 

Corollaries of the Binomial Theorem (concluded)

\[
\sum_{i=1}^{n} i \binom{n}{i} = n2^{n-1}.
\]

- Differentiate

\[(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i\]

to obtain

\[n(1 + x)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{i-1}.
\]

- Now set \(x = 1\).\(^a\)

\(^a\)An alternative proof to avoid calculus is to observe that \(i \binom{n}{i} = n \binom{n-1}{i-1}\). So \(\sum_{i=1}^{n} i \binom{n}{i} = \sum_{i=1}^{n} n \binom{n-1}{i-1} = n \sum_{i=1}^{n} \binom{n-1}{i-1} = n2^{n-1}\). Contributed by Mr. Kung-Ching Lin (B00703082) on February 27, 2012.
Binary Strings with Even Weight

- Consider a binary string $x_1 x_2 \cdots x_n$.  
  - The **weight** of $x_1 x_2 \cdots x_n$ is defined as $\sum_i x_i$.
- There are $2^n$ strings.
- Among them, 
  
  $\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i}$  

  have an even weight.
  - 1 occurs in $2i$ positions.
  - E.g., $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$, and 
    $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 64$.
- But Eq. (12) equals $2^{n-1}$ by Eq. (9) on p. 57.
Majority Decision

In a court with $2n + 1$ judges, in how many ways can a majority “yes” decision be handed down?

- Because any vote has a majority, consider cases when the majority vote “yes.”
- There are $\binom{2n+1}{i}$ ways such that $i$ judges vote “yes.”
- From Eq. (10) on p. 58, the desired answer is

$$\sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2^{2n}.$$
Ways To Merge Sets

What is the number of ways to merge members of

\{\{1\}, \{2\}, \ldots, \{n\}\}

to form

\{\{1, 2, \ldots, n\}\}

in \(n - 1\) steps?

- Each merge involves two members.
- For example, the number is 3 when \(n = 3\):

\[
\begin{align*}
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{1, 2\}, \{3\}\} \rightarrow \{\{1, 2, 3\}\}, \\
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{1, 3\}, \{2\}\} \rightarrow \{\{1, 2, 3\}\}, \\
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{2, 3\}, \{1\}\} \rightarrow \{\{1, 2, 3\}\}.
\end{align*}
\]
Ways To Merge Sets (continued)

- The 1st step begins with $n$ members.
- In general, the $i$th step begins with $n - i + 1$ members.
- There are
  \[
  \binom{n - i + 1}{2}
  \]
  ways to pick the two members.
Ways To Merge Sets (concluded)

- The desired number is thus

\[
\prod_{i=1}^{n-1} \binom{n-i+1}{2} = \binom{n}{2} \binom{n-1}{2} \cdots \binom{2}{2}
\]

\[
= \frac{n! (n-1)! \cdots 2!}{2^{n-1} (n-2)! (n-3)! \cdots 1!}
\]

\[
= \frac{n! (n-1)!}{2^{n-1}}.
\]
The Multinomial Theorem

Theorem 15

\[(x_1 + x_2 + \cdots + x_t)^n = \sum_{0 \leq n_1, n_2, \ldots, n_t \leq n} \frac{n!}{n_1! n_2! \cdots n_t!} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.\]

- Expand \((x_1 + x_2 + \cdots + x_t)^n\).
- Each term in the expansion must have the form

  \[(\text{coefficient}) \times x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t},\]

  where \(0 \leq n_1, n_2, \ldots, n_t \leq n\) and \(n_1 + n_2 + \cdots + n_t = n\).
The Proof (concluded)

• The coefficient of

\[ x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \]

equals the number of ways to pick \( n_1 \) \( x_1 \)'s, \( n_2 \) \( x_2 \)'s, and so on.

• By Eq. (2) on p. 16, there are

\[
\binom{n}{n_1, n_2, \ldots, n_t} \equiv \frac{n!}{n_1! n_2! \cdots n_t!}
\]

ways.
Coefficient of $a^2b^3c^2d^5$ in $(a + 2b - 3c + 2d + 5)^{16}$

- Make $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, and $x_5 = 5$ symbolically.

- The coefficient of $a^2(2b)^3(-3c)^2(2d)^55^4$ is

  $$\binom{16}{2,3,2,5,4} = \frac{16!}{2!3!2!5!4!} = 302,702,400$$

  by the multinomial theorem with $n = 16$.

- The desired coefficient is then

  $$302,702,400 \times 2^3 \times (-3)^2 \times 2^5 \times 5^4$$

  $$= 435,891,456,000,000.$$
Distinct Objects into Identical Containers

Corollary 16  There are \( \frac{(rn)!}{(r!)^n n!} \) ways to distribute \( rn \) distinct objects into \( n \) identical containers so that each container contains exactly \( r \) objects.

- Consider \( (x_1 + x_2 + \cdots + x_n)^{rn} \).
- Let \( x_i \) denote the containers (distinct, for now).
- Each object is associated with one \( x_1 + x_2 + \cdots + x_n \).
- It means an object can be assigned to one of the \( n \) containers.

- What does the coefficient of 
  \[ x_1^r x_2^r \cdots x_n^r \]
  mean?
Distinct Objects into Identical Containers (continued)

- Take \( n = 3 \) and \( r = 2 \).
- So we have

\[
(x_1 + x_2 + x_3)^6 = (x_1^6 + \cdots + x_3^6) \\
+ 6 (x_1^5 x_2 + \cdots + x_2 x_3^5) \\
+ 15 (x_1^4 x_2^2 + \cdots + x_2^2 x_3^4) \\
+ 20 (x_1^3 x_2^3 + \cdots + x_2^3 x_3^3) \\
+ 30 (x_1^4 x_2 x_3 + \cdots + x_1 x_2 x_3^4) \\
+ 60 (x_1^3 x_2^2 x_1 + \cdots + x_1 x_2^2 x_3^3) \\
+ 90 x_1^2 x_2^2 x_3^2.
\]
Distinct Objects into Identical Containers (concluded)

- It is the number of ways $rn$ distinct objects can be distributed into $n$ distinct containers, each of which contains $r$ objects.

- By Theorem 15 (p. 73), it is
  \[
  \binom{rn}{r, r, \ldots, r} = \frac{(rn)!}{r!r! \cdots r!}.
  \]

- Finally, divide the above count by $n!$ to remove the identities of the containers.

**Corollary 17** \( \frac{(rn)!}{(r!)^n n!} \) is an integer.

- Immediate from Corollary 16 (p. 76).
An Alternative Proof of Corollary 17 (p. 78)

\[
\frac{(rn)!}{(r!)^n n!} = \frac{1}{n!} \frac{(rn)!}{[r(n-1)]! r! [r(n-2)]! r! \cdots [r(n-n)]! r!}
= \prod_{k=0}^{n-1} \left( \frac{r(n-k)}{r} \right) \frac{n}{n!}
= \prod_{k=0}^{n-1} \frac{(r(n-k))}{n-k} = \prod_{k=0}^{n-1} \frac{[r(n-k)]!}{(n-k)r![r(n-k-1)]!}
= \prod_{k=0}^{n-1} \frac{r(n-k)[r(n-k)-1]!}{(n-k)r[r-1]![r(n-k-1)]!} = \prod_{k=0}^{n-1} \left( \frac{r(n-k)-1}{r-1} \right).
\]

\(^{\text{a}}\)Contributed by Mr. Ansel Lin (B93902003) on September 20, 2004.
Combinations with Repetition

**Theorem 18** Suppose there are \( n \) distinct objects and \( r \geq 0 \) is an integer. The number of selections of \( r \) of these objects, with repetition, is

\[
C(n + r - 1, r) = \binom{n + r - 1}{r}.
\]

- Note that the order of selection is not important.
- Imagine there are \( n \) distinct types of objects.
The Proof (continued)

- Permute

\[
\underbrace{xx \cdots x}_{n-1} \bigg| \bigg| \underbrace{x}_{r} \bigg| \bigg| \bigg|
\]

- Think of the \(i\)th interval as containing the \(i\)th type of objects.

- So

\[
x x | xxx | x | \big| | | \big|
\]

means, out of 7 distinct objects, we pick 2 type-1 objects, 3 type-2 objects, and 1 type-3 object.
The Proof (concluded)

- Our goal equals the number of permutations of \( \underbrace{xx \cdots x}_{r} \underbrace{\mid \cdots \mid}_{n-1} \).

- By Eq. (2) on p. 16, it is
  \[
  \frac{(r + n - 1)!}{r!(n - 1)!} = \binom{n + r - 1}{r} = C(n + r - 1, r).
  \]
Combinatorial Proof of Corollary 6 (p. 34)\textsuperscript{a}

**Corollary 19** For \( m, n \geq 0 \), \( \sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m} \).

- The number of ways to select \( m \) objects out of \( n + 2 \) types is \( \binom{n+m+1}{m} \) by Theorem 18 (p. 80).
- Alternatively, let us focus on how the objects of the first \( n + 1 \) types are chosen.
- There are \( \binom{n+m}{m} \) ways to select \( m \) objects out of the first \( n + 1 \) types.
- There are \( \binom{n+m-1}{m-1} \) ways to select \( m - 1 \) objects out of the first \( n + 1 \) types and 1 object out of the last type.

\textsuperscript{a}Contributed by Mr. Jerry Lin (B01902113) on March 13, 2014.
The Proof (concluded)

- There are \( \binom{n+m-2}{m-2} \) ways to select \( m - 2 \) objects out of the first \( n + 1 \) types and 2 objects of the last type.
- ....
- By induction,

\[
\binom{n + m + 1}{m} = \binom{n + m}{m} + \binom{n + m - 1}{m - 1} + \binom{n + m - 2}{m - 2} + \cdots + \binom{n + 0}{0}.
\]
Integer Solutions of a Linear Equation

The following three problems are equivalent:

1. The number of nonnegative integer solutions of

   \[ x_1 + x_2 + \cdots + x_n = r. \]

2. The number of selections, with repetition, of size \( r \) from
   a collection of \( n \) distinct objects (Theorem 18 on p. 80).

3. The number of ways \( r \) identical objects can be
   distributed among \( n \) distinct containers.\(^a\)

They all equal \( \binom{n+r-1}{r}. \)^b

\(^a\)The case of distinct objects and identical containers will be covered
on p. 257 (see p. 76 for a special case).

\(^b\)See p. 459 and p. 463 for alternative proofs.
Application: The Multinomial Theorem (p. 73)

- It concerns the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of
  $$(x_1 + x_2 + \cdots + x_t)^n.$$  
- Each term has the form $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ such that
  \begin{itemize}
  \item $n_1 + n_2 + \cdots + n_t = n$, and
  \item $0 \leq n_1, n_2, \ldots, n_t$.
  \end{itemize}
- How many distinct forms are there?
- For example, consider
  $$x_1 + x_2 + x_3 = 2.$$
Application: The Multinomial Theorem (continued)

- Now,

\[(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.\]

- E.g., the solution “\(x_1 = 1, x_2 = 1, x_3 = 0\)” to \(x_1 + x_2 + x_3 = 2\) contributes to the term \(x_1^1x_2^1x_3^0 = x_1x_2\).

- So there are 6 nonnegative integer solutions to \(x_1 + x_2 + x_3 = 2\) because there are 6 terms.
Application: The Multinomial Theorem (concluded)

- The desired number of terms is therefore

\[
\binom{n + t - 1}{n},
\]

from the equivalencies on p. 85.

- Indeed, \(\binom{2+3-1}{2} = 6\).
Positive Integer Solutions of a Linear Equation

• Consider

\[ x_1 + x_2 + \cdots + x_n = r, \]

where \( x_i > 0 \) for \( 1 \leq i \leq n \).

• Define \( x'_i \equiv x_i - 1 \).

• The original problem becomes

\[ x'_1 + x'_2 + \cdots + x'_n = r - n, \]

where \( x'_i \geq 0 \) for \( 1 \leq i \leq n \).

• The number of solutions is therefore (p. 85)

\[
\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n} = \binom{r - 1}{n - 1}.
\] (13)
Application: Subsets with Restrictions

How many $n$-element subsets of $\{1, 2, \ldots, r\}$ contain no consecutive integers?

- Say $r = 4$ and $n = 2$.
- Then the valid 2-element subsets of $\{1, 2, 3, 4\}$ are

$\{1, 3\}, \{1, 4\}, \{2, 4\}$. 
The Proof (continued)

- For each valid subset \( \{i_1, i_2, \ldots, i_n\} \), where \( 1 \leq i_1 < i_2 < \cdots < i_n \leq r \), define
  \[
  d_k = i_{k+1} - i_k.
  \]

- As “placeholders,” introduce
  \[
  i_0 = 1, \\
  i_{n+1} = r.
  \]

- Then, by telescoping,
  \[
  d_0 + d_1 + \cdots + d_n = i_{n+1} - i_0 = r - 1.
  \]
The Proof (continued)

- Observe that

\[ 0 \leq d_0, d_n \]
\[ 2 \leq d_1, d_2, \ldots, d_{n-1}. \]

- Define

\[ d'_0 \equiv d_0, \]
\[ d'_i \equiv d_i - 2, \quad i = 1, 2, \ldots, n - 1, \]
\[ d'_n \equiv d_n. \]
The Proof (concluded)

- So equivalently,

\[ d'_0 + d'_1 + \cdots + d'_n = r - 1 - 2(n - 1) \]

with \( 0 \leq d'_0, d'_1, \ldots, d'_n \).

- The answer to the desired number is (p. 85)

\[
\begin{pmatrix}
(n + 1) + (r - 1 - 2(n - 1)) - 1 \\
r - 1 - 2(n - 1)
\end{pmatrix}

= \begin{pmatrix}
r - n + 1 \\
r - 2n + 1
\end{pmatrix}

= \begin{pmatrix}
r - n + 1 \\
n
\end{pmatrix}.

(14)
Application: Political Majority

In how many ways can $2n + 1$ seats in a parliament be divided among 3 parties so that the coalition of any 2 parties form a majority?

- If $n = 2$, there are 5 seats.
- Clearly, no party should have 3 or more seats.
- The only valid distribution of the 5 seats to 3 parties is: 2, 2, 1.
- The number of ways is therefore 3.
The Proof (continued)

- This is a problem of distributing identical objects (the seats) among distinct containers (the parties) (p. 85).

- So without the majority condition, the number is 
  \[ \binom{3+(2n+1)-1}{2n+1} = \binom{2n+3}{2}. \]

- Observe that the majority condition is violated if and only if a party gets \( n + 1 \) or more seats (why?).
The Proof (concluded)

- If a given party gets $n + 1$ or more seats, the number of ways of distributing the seats is

$$\binom{3 + n - 1}{n} = \binom{n + 2}{2}.$$ 

- Allocate $n + 1$ seats to that party before allocating the remaining $n$ seats to the 3 parties.

- Then refer to p. 85 for the formula.

- The desired number of no dominating party is

$$\binom{2n + 3}{2} - 3\binom{n + 2}{2} = \frac{n}{2} (n + 1) = \binom{n + 1}{2}. \quad (15)$$
Political Majority: An Alternative Proof\textsuperscript{a}

- Each party can hold up to \( n \) seats.\textsuperscript{b}
- Give each party \( n \) slots to hold real seats.
- There will be

\[
3n - (2n + 1) = n - 1
\]

empty slots in the end.

\textsuperscript{a}Contributed by Mr. Weicheng Lee (B01902065) on March 14, 2013.

\textsuperscript{b}The majority condition holds if and only if no party gets \( n + 1 \) or more seats.
Political Majority: An Alternative Proof (concluded)

- So the answer to the desired number is the number of ways to distribute the \( n - 1 \) empty slots to 3 parties.
- The count is (p. 85)

\[
\binom{3 + (n - 1) - 1}{n - 1} = \binom{n + 1}{n - 1} = \binom{n + 1}{2}.
\]
Integer Solutions of a Linear Inequality

• Consider

\[ x_1 + x_2 + \cdots + x_n \leq r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n \).

• It is equivalent to

\[ x_1 + x_2 + \cdots + x_n + x_{n+1} = r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n + 1 \).

• The number of integer solutions of the original inequality is therefore (p. 85)

\[
\binom{(n + 1) + r - 1}{r} = \binom{n + r}{r}. \quad (16)
\]
Integer Solutions of a Strict Linear Inequality

- Consider

\[ x_1 + x_2 + \cdots + x_n < r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n \).

- It is equivalent to

\[ x_1 + x_2 + \cdots + x_n \leq r - 1, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n \).

- By Eq. (16) on p. 99, the number of nonnegative integer solutions is

\[
\binom{n + r - 1}{r - 1}.
\]