Problem 1 (10 points) Prove that any two consecutive Fibonacci numbers are relatively prime. The Fibonacci recurrence equation is $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$ and $F_1 = 1$.

Ans: Since $\gcd(F_0, F_1) = \gcd(F_1, F_2) = 1$, consider $n > 2$. Inductively, assume $\gcd(F_i, F_{i+1}) = 1$ for all $i < n$. Suppose $\gcd(F_n, F_{n+1}) = d > 1$. In particular, $d$ divides $F_n$ and $F_{n+1}$. Then $F_{n+1} - F_n = F_{n-1}$ is also divisible by $d$. After the above, $d$ divides $\gcd(F_{n-1}, F_n)$, contradicting the induction step. Thus $F_n$ and $F_{n+1}$ must be relatively prime.

Problem 2 (10 points) Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 4a_n = 0$, where $n \geq 0$ and $a_0 = 4$, $a_1 = 13$.

Ans: $a_n = \pm \sqrt{51(4^n) - 35}, n \geq 0$.

Problem 3 (10 points) If $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, and $a_3 = 37$ satisfy the recurrence relation $a_{n+2} + ba_{n+1} + ca_n = 0$, where $n \geq 0$ and $b$, $c$ are constants, determine $b$, $c$ and solve for $a_n$.

Ans: $b = -4$, $c = -21$, $a_n = (1/10)[7^n - (-3)^n], n \geq 0$.

Problem 4 (5 points) Can a simple graph exist with 15 vertices each of degree five?

Ans: No.
**Problem 5 (10 points)** If the simple graph $G$ has $v$ vertices and $e$ edges, how many edges does $G$ have?

**Ans:** $v(v - 1)/2 - e$. 

**Problem 6 (10 points)** Prove that an acyclic digraph has at least one node of out-degree zero. (An acyclic digraph is a directed graph containing no directed cycles.)

**Ans:** Consider the last node of any longest path in the digraph. This node can have no nodes that are incident from it because; otherwise, there would be a cycle.

**Problem 7 (10 points)** If $G = (V, E)$ is a loop-free undirected graph, prove that $G$ is a tree if there is a unique path between any two vertices of $G$.

**Ans:** If there is a unique path between each pair of vertices in $G$, then $G$ is connected. If $G$ contains a cycle, then there is a pair of vertices $x, y$ with two distinct paths connecting $x$ and $y$. Hence, $G$ is a loop-free connected undirected graph with no cycles, so a tree.

**Problem 8 (5 points)** Give an example of an undirected graph $G = (V, E)$, where $|V| = |E| + 1$ but $G$ is not a tree.

**Ans:**

```
\begin{center}
\begin{tikzpicture}
    \node[vertex] (a) at (0,0) {$a$};
    \node[vertex] (b) at (1,1) {$b$};
    \node[vertex] (c) at (1,-1) {$c$};
    \node[vertex] (d) at (2,0) {$d$};
    \draw (a) -- (b) -- (c) -- (a);
    \draw (c) -- (d);
\end{tikzpicture}
\end{center}
```
**Problem 9 (10 points)** If $G$ is a group, let $H = \{a \in G \mid ag = ga \text{ for all } g \in G\}$. Prove that $H$ is a subgroup of $G$. (The subgroup $H$ is called the center of $G$.)

**Ans:** Since $ag = ga$ for all $g \in G$, it follows that $a \in H$ and $H \neq \emptyset$. If $x, y \in H$, then $xg = gx$ and $yg = gy$ for all $g \in G$. Consequently, $(xy)g = x(yg) = x(gy) = (xg)y = g(xy)$ for all $g \in G$, and we have $xy \in H$. Finally, for each $x \in H, g \in G, xg^{-1} = g^{-1}x$. So $(xg^{-1})^{-1} = (g^{-1}x)^{-1}$, or $gx^{-1} = x^{-1}g$, and $x^{-1} \in H$. Therefore, $H$ is a subgroup of $G$.

**Problem 10 (10 points)** Verify that $(\mathbb{Z}_{p}, \cdot)$ is cyclic for the primes $p = 7$ and $11$.

**Ans:** $\mathbb{Z}_{7}^{*} = \langle 3 \rangle = \langle 5 \rangle$; $\mathbb{Z}_{11}^{*} = \langle 2 \rangle = \langle 6 \rangle = \langle 7 \rangle = \langle 8 \rangle$.

**Problem 11 (10 points)** Prove that every group of prime order is cyclic.

**Ans:** Pick any element $a \neq e$ of the group $G$. As $o(a)$ divides $|G|$, a prime number, $o(a) = |G|$. This implies that every $b \in G$ must be of the form $a^k$ for some $k \in \mathbb{Z}$.