Magic 3/4?

- The number 3/4 bounds the probability (ratio) of a right answer away from 1/2.
- Any constant *strictly* between 1/2 and 1 can be used without affecting the class BPP.
- In fact, as with RP,

$$\frac{1}{2}+\frac{1}{q(n)}$$

  for any polynomial $q(n)$ can replace 3/4.
- The next algorithm shows why.

The Majority Vote Algorithm

Suppose L is decided by N by majority $(1/2) + \epsilon$.

for $i = 1, 2, \ldots, 2k + 1$ **do**
    Run N on input x;
end for
if "yes" is the majority answer **then**
    "yes";
else
    "no";
end if

Analysis

- By Corollary 77 (p. 604), the probability of a false answer is at most $e^{-\epsilon^2 k}$.
- By taking $k = \lceil 2/\epsilon^2 \rceil$, the error probability is at most 1/4.
- Even if ϵ is any inverse polynomial, k remains a polynomial in n.
- The running time remains polynomial: $2k+1$ times N's running time.

Aspects of BPP

- BPP is the most comprehensive yet plausible notion of efficient computation.
  - If a problem is in BPP, we take it to mean that the problem can be solved efficiently.
  - In this aspect, BPP has effectively replaced P.
- $(\text{RP} \cup \text{coRP}) \subseteq (\text{NP} \cup \text{coNP})$.
- $(\text{RP} \cup \text{coRP}) \subseteq \text{BPP}$.
- Whether $\text{BPP} \subseteq (\text{NP} \cup \text{coNP})$ is unknown.
- But it is unlikely that $\text{NP} \subseteq \text{BPP}$.[a]

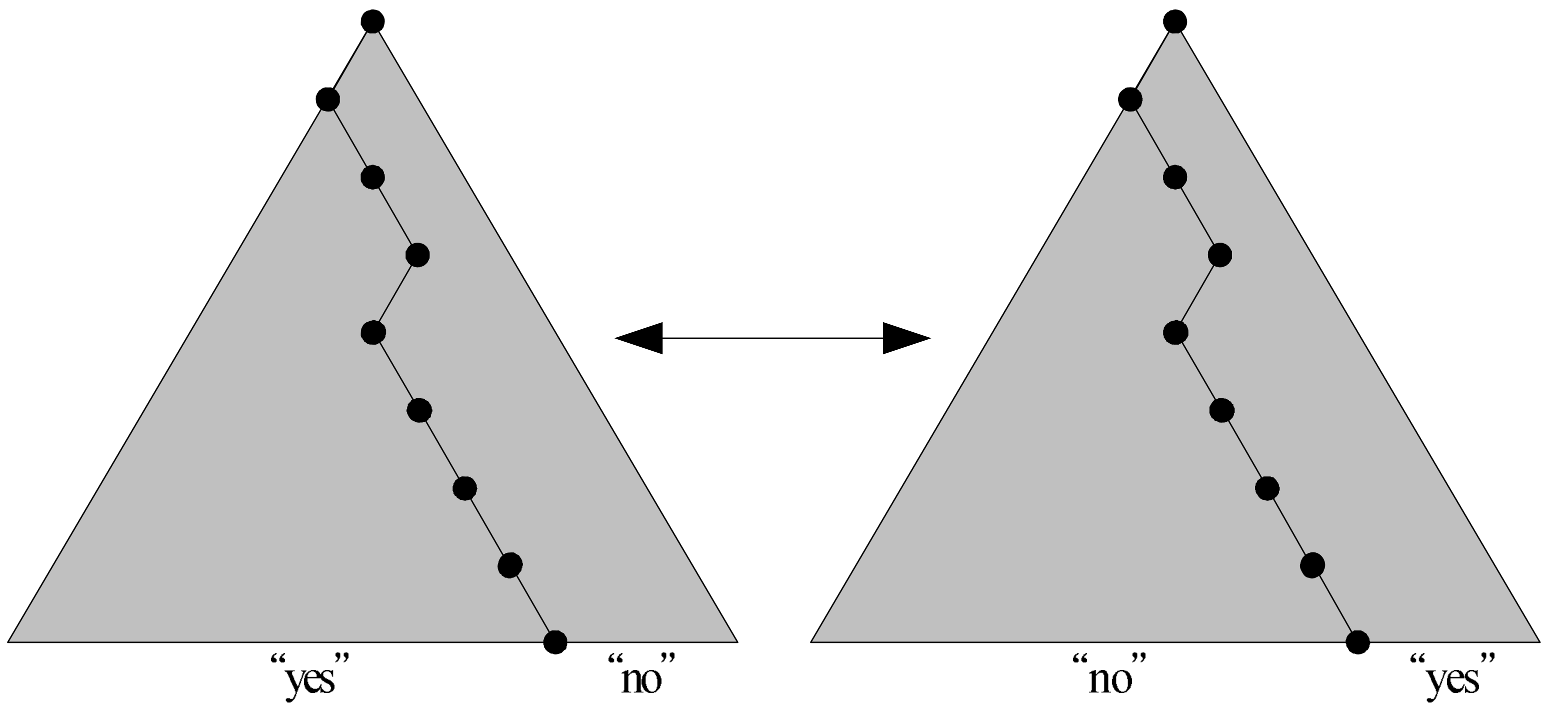
[a] See p. 621.

coBPP

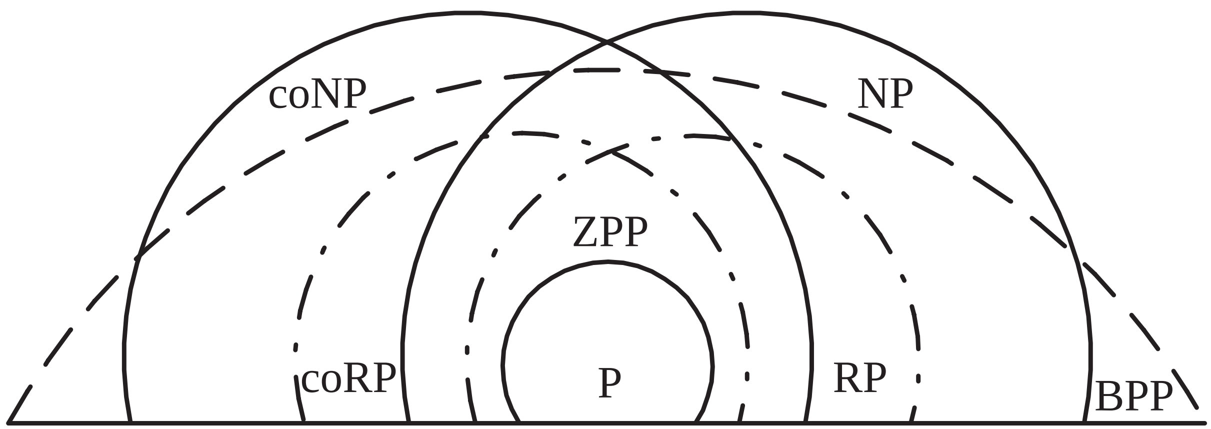
- The definition of BPP is symmetric: acceptance by clear majority and rejection by clear majority.
- An algorithm for $L \in \text{BPP}$ becomes one for $\bar{L}$ by reversing the answer.
- So $\bar{L} \in \text{BPP}$ and $\text{BPP} \subseteq \text{coBPP}$.
- Similarly $\text{coBPP} \subseteq \text{BPP}$.
- Hence $\text{BPP} = \text{coBPP}$.
- This approach does not work for RP.[a]

[a] It did not work for NP either.

BPP and coBPP



"The Good, the Bad, and the Ugly"



Circuit Complexity

- Circuit complexity is based on boolean circuits instead of Turing machines.
- A boolean circuit with n inputs computes a boolean function of n variables.
- Now, identify `true`/1 with "yes" and `false`/0 with "no."
- Then a boolean circuit with n inputs accepts certain strings in $\{0, 1\}^n$.
- To relate circuits with an arbitrary language, we need one circuit for each possible input length n.

Formal Definitions

- The **size** of a circuit is the number of *gates* in it.
- A **family of circuits** is an infinite sequence $\mathcal{C} = (C_0, C_1, \ldots)$ of boolean circuits, where C_n has n boolean inputs.
- For input $x \in \{0,1\}^*$, $C_{|x|}$ outputs 1 if and only if $x \in L$.
- In other words,

$$C_n \text{ accepts } L \cap \{0,1\}^n.$$

Formal Definitions (concluded)

- $L \subseteq \{0,1\}^*$ has **polynomial circuits** if there is a family of circuits $\mathcal{C}$ such that:
  - The size of C_n is at most $p(n)$ for some fixed polynomial p.
  - C_n accepts $L \cap \{0,1\}^n$.

Exponential Circuits Suffice for All Languages

- Theorem 16 (p. 208) implies that there are languages that cannot be solved by circuits of size $2^n/(2n)$.
- But surprisingly, circuits of size 2^{n+2} can solve *all* problems, decidable or otherwise!

Exponential Circuits Suffice for All Languages (continued)

Proposition 78 *All decision problems (decidable or otherwise) can be solved by a circuit of size* 2^{n+2}.

- We will show that for any language $L \subseteq \{0,1\}^*$, $L \cap \{0,1\}^n$ can be decided by a circuit of size 2^{n+2}.
- Define boolean function $f : \{0,1\}^n \to \{0,1\}$, where

$$f(x_1 x_2 \cdots x_n) = \begin{cases} 1 & x_1 x_2 \cdots x_n \in L, \\ 0 & x_1 x_2 \cdots x_n \notin L. \end{cases}$$

The Proof (concluded)

- Clearly, any circuit that implements f decides $L \cap \{0, 1\}^n$.
- Now,

$$f(x_1 x_2 \cdots x_n) = (x_1 \wedge f(1 x_2 \cdots x_n)) \vee (\neg x_1 \wedge f(0 x_2 \cdots x_n)).$$

- The circuit size $s(n)$ for $f(x_1 x_2 \cdots x_n)$ hence satisfies

$$s(n) = 4 + 2s(n-1)$$

  with $s(1) = 1$.
- Solve it to obtain $s(n) = 5 \times 2^{n-1} - 4 \leq 2^{n+2}$.

The Circuit Complexity of P

Proposition 79 *All languages in P have polynomial circuits.*

- Let $L \in \mathrm{P}$ be decided by a TM in time $p(n)$.
- By Corollary 34 (p. 312), there is a circuit with $O(p(n)^2)$ gates that accepts $L \cap \{0, 1\}^n$.
- The size of that circuit depends only on L and the length of the input.
- The size of that circuit is polynomial in n.

Polynomial Circuits vs. P

- Is the converse of Proposition 79 true?
  - Do polynomial circuits accept only languages in P?
- No.
- Polynomial circuits can accept *undecidable* languages!

BPP's Circuit Complexity: Adleman's Theorem

Theorem 80 (Adleman, 1978) *All languages in BPP have polynomial circuits.*

- Our proof will be *nonconstructive* in that only the existence of the desired circuits is shown.
  - Recall our proof of Theorem 16 (p. 208).
  - Something exists if its probability of existence is nonzero.
- It is not known how to efficiently generate circuit C_n.
  - If the construction of C_n can be made efficient, then P = BPP, an unlikely result.

The Proof

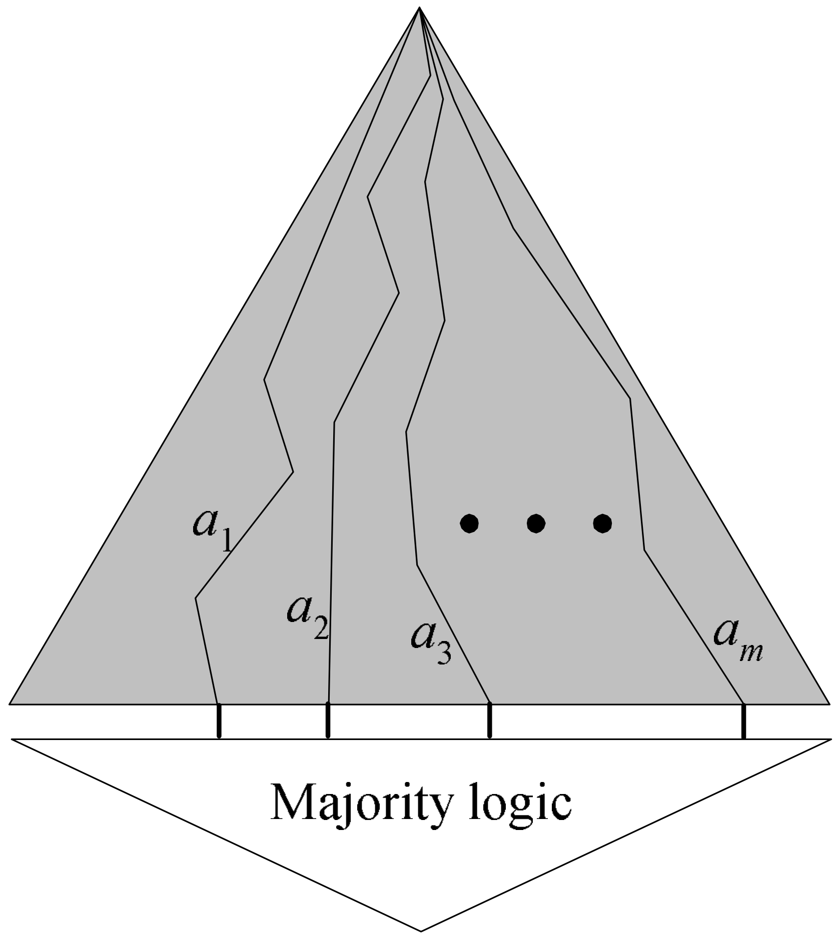
- Let $L \in \text{BPP}$ be decided by a precise polynomial-time NTM N by clear majority.
- We shall prove that L has polynomial circuits $C_0, C_1, \ldots$.
  - These *deterministic* circuits do not err.
- Suppose N runs in time $p(n)$, where $p(n)$ is a polynomial.
- Let $A_n = \{ a_1, a_2, \ldots, a_m \}$, where $a_i \in \{ 0, 1 \}^{p(n)}$.
- Each $a_i \in A_n$ represents a sequence of nondeterministic choices (i.e., a computation path) for N.
- Pick $m = 12(n + 1)$.

The Proof (continued)

- Let x be an input with $|x| = n$.
- Circuit C_n simulates N on x with all sequences of choices in A_n and then takes the majority of the m outcomes.[a]
  - Note that each A_n yields a circuit.
- As N with a_i is a polynomial-time deterministic TM, it can be simulated by polynomial circuits of size $O(p(n)^2)$.
  - See the proof of Proposition 79 (p. 619).

[a] As m is even, there may be no clear majority. Still, the probability of that happening is very small and does not materially affect our general conclusion. Thanks to a lively class discussion on December 14, 2010.

The Circuit



The Proof (continued)

- The size of C_n is therefore $O(mp(n)^2) = O(np(n)^2)$.
  - This is a polynomial.
- We now confirm the existence of an A_n making C_n correct on *all* n-bit inputs.
- Call a_i **bad** if it leads N to an error (a false positive or a false negative) for x.
- Select A_n uniformly randomly.

The Proof (continued)

- For each $x \in \{0, 1\}^n$, $1/4$ of the computations of N are erroneous.
- Because the sequences in A_n are chosen randomly and independently, the expected number of bad a_i's is $m/4$.[a]
- Also note after fixing the input x, the circuit is a function of the random bits.

[a] So the proof will not work for NP. Contributed by Mr. Ching-Hua Yu (`D00921025`) on December 11, 2012.

The Proof (continued)

- By the Chernoff bound (p. 599), the probability that the number of bad a_i's is $m/2$ or more is at most

$$e^{-m/12} < 2^{-(n+1)}.$$

- The error probability of using the majority rule is thus

$$< 2^{-(n+1)}$$

for each $x \in \{ 0, 1 \}^n$.

The Proof (continued)

- The probability that there is an x such that A_n results in an incorrect answer is

$$< 2^n 2^{-(n+1)} = 2^{-1}.$$

  - Recall the union bound (**Boole's inequality**): $\text{prob}[\, A \cup B \cup \cdots] \leq \text{prob}[\, A\,] + \text{prob}[\, B\,] + \cdots$.

- We just showed that at least half of them are correct.

- So with probability ≥ 0.5, a random A_n produces a correct C_n for *all* inputs of length n.

  - Of course, verifying this fact may take a long time.

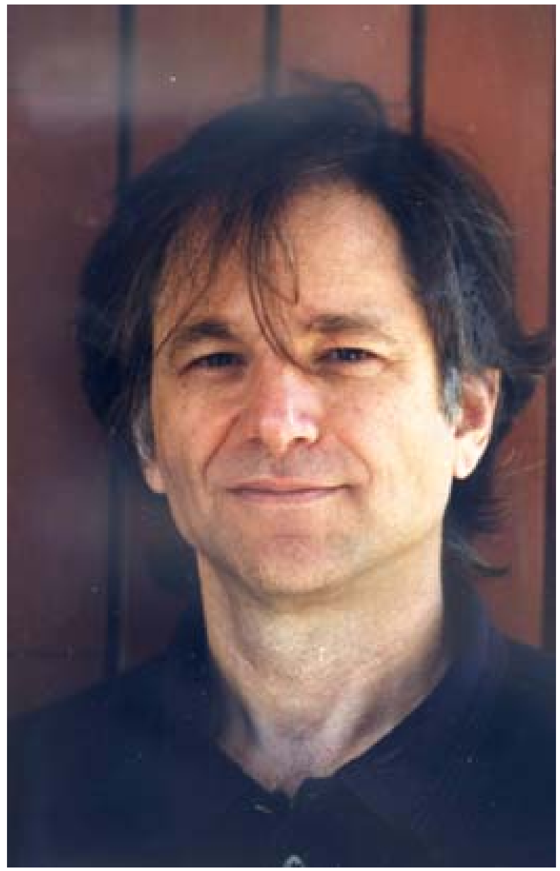
The Proof (concluded)

- Because this probability exceeds 0, an A_n that makes majority vote work for all inputs of length n exists.
- Hence a correct C_n exists.[a]
- We have used the **probabilistic method** popularized by Erdős.[b]
- This result answers the question on p. 530 with a "yes."

[a] Quine (1948), "To be is to be the value of a bound variable."

[b] A counting argument in the probabilistic language.

Leonard Adleman[a] (1945–)



[a]Turing Award (2002).

Paul Erdős (1913–1996)

Cryptography

Whoever wishes to keep a secret
must hide the fact that he possesses one.
— Johann Wolfgang von Goethe (1749–1832)

Cryptography

- **Alice** (A) wants to send a message to **Bob** (B) over a channel monitored by **Eve** (eavesdropper).
- The protocol should be such that the message is known only to Alice and Bob.
- The art and science of keeping messages secure is **cryptography**.

$$\text{Alice} \xrightarrow{\text{Eve}} \text{Bob}$$

Encryption and Decryption

- Alice and Bob agree on two algorithms E and D—the **encryption** and the **decryption algorithms**.
- Both E and D are known to the public in the analysis.
- Alice runs E and wants to send a message x to Bob.
- Bob operates D.

Encryption and Decryption (concluded)

- Privacy is assured in terms of two numbers e, d, the **encryption** and **decryption keys**.
- Alice sends $y = E(e, x)$ to Bob, who then performs $D(d, y) = x$ to recover x.
- x is called **plaintext**, and y is called **ciphertext**.[a]

[a] Both "zero" and "cipher" come from the same Arab word.

Some Requirements

- D should be an inverse of E given e and d.
- D and E must both run in (probabilistic) polynomial time.
- Eve should not be able to recover x from y without knowing d.
  - As D is public, d must be kept secret.
  - e may or may not be a secret.

Degree of Security

- **Perfect secrecy**: After a ciphertext is intercepted by the enemy, the a posteriori probabilities of the plaintext that this ciphertext represents are identical to the a priori probabilities of the same plaintext before the interception.
  - The probability that plaintext $\mathcal{P}$ occurs is independent of the ciphertext $\mathcal{C}$ being observed.
  - So knowing $\mathcal{C}$ yields no advantage in recovering $\mathcal{P}$.

Degree of Security (concluded)

- Such systems are said to be **informationally secure**.
- A system is **computationally secure** if breaking it is theoretically possible but computationally infeasible.

Conditions for Perfect Secrecy[a]

- Consider a cryptosystem where:
  - The space of ciphertext is as large as that of keys.
  - Every plaintext has a nonzero probability of being used.
- It is **perfectly secure** if and only if the following hold.
  - A key is chosen with uniform distribution.
  - For each plaintext x and ciphertext y, there exists a unique key e such that $E(e, x) = y$.

[a] Shannon (1949).

The One-Time Pad[a]

1: Alice generates a random string r as long as x;

2: Alice sends r to Bob over a secret channel;

3: Alice sends $x \oplus r$ to Bob over a public channel;

4: Bob receives y;

5: Bob recovers $x := y \oplus r$;

---

[a] Mauborgne & Vernam (1917); Shannon (1949). It was allegedly used for the hotline between Russia and U.S.

Analysis

- The one-time pad uses $e = d = r$.
- This is said to be a **private-key cryptosystem**.
- Knowing x and knowing r are equivalent.
- Because r is random and private, the one-time pad achieves perfect secrecy.[a]
- The random bit string must be new for each round of communication.
- But the assumption of a private channel is problematic.

[a] See p. 640.

Public-Key Cryptography[a]

- Suppose only d is private to Bob, whereas e is public knowledge.
- Bob generates the (e, d) pair and publishes e.
- Anybody like Alice can send $E(e, x)$ to Bob.
- Knowing d, Bob can recover x via

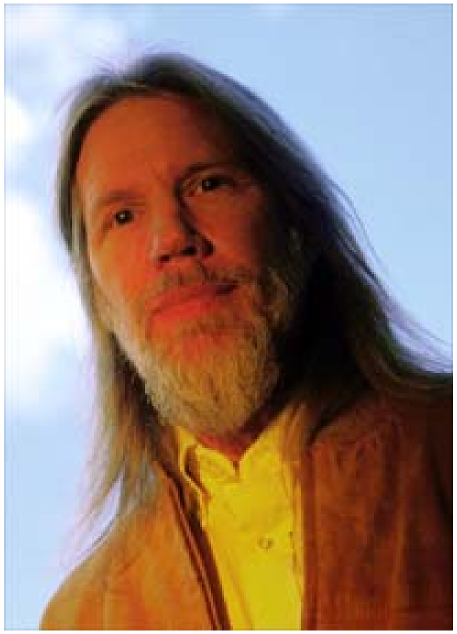
$$D(d, E(e, x)) = x.$$

[a] Diffie & Hellman (1976).

Public-Key Cryptography (concluded)

- The assumptions are complexity-theoretic.
  - It is computationally difficult to compute d from e.
  - It is computationally difficult to compute x from y without knowing d.

Whitfield Diffie[a] (1944–)



[a]Turing Award (2016).

Martin Hellman[a] (1945–)

[a]Turing Award (2016).

Complexity Issues

- Given y and x, it is easy to verify whether $E(e, x) = y$.
- Hence one can always guess an x and verify.
- Cracking a public-key cryptosystem is thus in NP.
- A *necessary* condition for the existence of secure public-key cryptosystems is $\text{P} \neq \text{NP}$.
- But more is needed than $\text{P} \neq \text{NP}$.
- For instance, it is not sufficient that D is hard to compute in the *worst* case.
- It should be hard in "most" or "average" cases.

One-Way Functions

A function f is a **one-way function** if the following hold.[a]

1. f is one-to-one.
2. For all $x \in \Sigma^*$, $|\,x\,|^{1/k} \leq |f(x)| \leq |\,x\,|^k$ for some $k > 0$.
   - f is said to be **honest**.
3. f can be computed in polynomial time.
4. f^{-1} cannot be computed in polynomial time.
   - Exhaustive search works, but it must be slow.

[a] Diffie & Hellman (1976); Boppana & Lagarias (1986); Grollmann & Selman (1988); Ko (1985); Ko, Long, & Du (1986); Watanabe (1985); Young (1983).

Existence of One-Way Functions (OWFs)

- Even if P $\neq$ NP, there is no guarantee that one-way functions exist.
- No functions have been proved to be one-way.
- Is breaking glass a one-way function?

Candidates of One-Way Functions

- Modular exponentiation $f(x) = g^x \bmod p$, where g is a primitive root of p.
  - **Discrete logarithm** is hard.[a]
- The RSA[b] function $f(x) = x^e \bmod pq$ for an odd e relatively prime to $\phi(pq)$.
  - Breaking the RSA function is hard.

---

[a] Conjectured to be 2^{n^ϵ} for some $\epsilon > 0$ in both the worst-case sense and average sense. Doable in time $n^{O(\log n)}$ for finite fields of small characteristic (Barbulescu, et al., 2013). It is in NP in some sense (Grollmann & Selman, 1988).

[b] Rivest, Shamir, & Adleman (1978).

Candidates of One-Way Functions (concluded)

- Modular squaring $f(x) = x^2 \bmod pq$.
  - Determining if a number with a Jacobi symbol 1 is a quadratic residue is hard—the **quadratic residuacity assumption** (**QRA**).[a]
  - Breaking it is as hard as factorization when $p \equiv q \equiv 3 \bmod 4$.[b]

---

[a] Due to Gauss.
[b] Rabin (1979).

The Secret-Key Agreement Problem

- Exchanging messages securely using a private-key cryptosystem requires Alice and Bob have the *same* key.[a]
  - An example is the r in the one-time pad.[b]
- How can they agree on the same secret key when the channel is insecure?
- This is called the **secret-key agreement problem**.
- It was solved by Diffie and Hellman (1976) using one-way functions.

---

[a]See p. 642.
[b]See p. 641.

The Diffie-Hellman Secret-Key Agreement Protocol

1: Alice and Bob agree on a large prime p and a primitive root g of p; $\{p$ and g are public.$\}$

2: Alice chooses a large number a at random;

3: Alice computes $\alpha = g^a \bmod p$;

4: Bob chooses a large number b at random;

5: Bob computes $\beta = g^b \bmod p$;

6: Alice sends α to Bob, and Bob sends β to Alice;

7: Alice computes her key $\beta^a \bmod p$;

8: Bob computes his key $\alpha^b \bmod p$;

Analysis

- The keys computed by Alice and Bob are identical as

$$\beta^a = g^{ba} = g^{ab} = \alpha^b \bmod p.$$

- To compute the common key from p, g, α, β is known as the **Diffie-Hellman problem**.
- It is conjectured to be hard.[a]
- If discrete logarithm is easy, then one can solve the Diffie-Hellman problem.
  - Because a and b can then be obtained by Eve.
- But the other direction is still open.

---

[a]This is the **computational Diffie-Hellman assumption** (CDH).

The RSA Function

- Let p, q be two distinct primes.
- The RSA function is $x^e \bmod pq$ for an odd e relatively prime to $\phi(pq)$.
  - By Lemma 58 (p. 480),
$$\phi(pq) = pq\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) = pq - p - q + 1. \quad (15)$$
- As $\gcd(e, \phi(pq)) = 1$, there is a d such that
$$ed \equiv 1 \bmod \phi(pq),$$
which can be found by the Euclidean algorithm.[a]

[a] One can think of d as e^{-1}.

A Public-Key Cryptosystem Based on RSA

- Bob generates p and q.
- Bob publishes pq and the encryption key e, a number relatively prime to $\phi(pq)$.
  - The encryption function is
    $$y = x^e \bmod pq.$$
  - Bob calculates $\phi(pq)$ by Eq. (15) (p. 655).
  - Bob then calculates d such that $ed = 1 + k\phi(pq)$ for some $k \in \mathbb{Z}$.

A Public-Key Cryptosystem Based on RSA (continued)

- The decryption function is

$$y^d \bmod pq.$$

- It works because

$$y^d = x^{ed} = x^{1+k\phi(pq)} = x \bmod pq$$

by the Fermat-Euler theorem when $\gcd(x, pq) = 1$ (p. 489).

A Public-Key Cryptosystem Based on RSA (continued)

- What if x is not relatively prime to pq?[a]
- As $\phi(pq) = (p-1)(q-1)$,

$$ed = 1 + k(p-1)(q-1).$$

- Say $x \equiv 0 \bmod p$.
- Then

$$y^d \equiv x^{ed} \equiv 0 \equiv x \bmod p.$$

[a] Of course, one would be unlucky here.

A Public-Key Cryptosystem Based on RSA (continued)

- On the other hand, either $x \not\equiv 0 \bmod q$ or $x \equiv 0 \bmod q$.
- If $x \not\equiv 0 \bmod q$, then

$$\begin{aligned} y^d &\equiv x^{ed} \equiv x^{ed-1} x \equiv x^{k(p-1)(q-1)} x \equiv \left(x^{q-1}\right)^{k(p-1)} x \\ &\equiv x \bmod q. \end{aligned}$$

  by Fermat's "little" theorem (p. 487).
- If $x \equiv 0 \bmod q$, then

$$y^d \equiv x^{ed} \equiv 0 \equiv x \bmod q.$$

A Public-Key Cryptosystem Based on RSA (concluded)

- By the Chinese remainder theorem (p. 486),

$$y^d \equiv x^{ed} \equiv 0 \equiv x \bmod pq,$$

even when x is not relatively prime to p.

- When x is not relatively prime to q, the same conclusion holds.

The "Security" of the RSA Function

- Factoring pq or calculating d from (e, pq) seems hard.[a]
- Breaking the last bit of RSA is as hard as breaking the RSA.[b]
- Recommended RSA key sizes:[c]
  - 1024 bits up to 2010.
  - 2048 bits up to 2030.
  - 3072 bits up to 2031 and beyond.

---

[a] See also p. 485.

[b] Alexi, Chor, Goldreich, & Schnorr (1988).

[c] RSA (2003). RSA was acquired by EMC in 2006 for 2.1 billion US dollars.

The "Security" of the RSA Function (continued)

- Recall that problem A is "harder than" problem B if solving A results in solving B.
  - Factorization is "harder than" breaking the RSA.
  - It is not hard to show that calculating Euler's phi function[a] is "harder than" breaking the RSA.
  - Factorization is "harder than" calculating Euler's phi function (see Lemma 58 on p. 480).
  - So factorization is harder than calculating Euler's phi function, which is harder than breaking the RSA.

[a]When the input is not factorized!

The "Security" of the RSA Function (concluded)

- Factorization cannot be NP-hard unless NP = coNP.[a]
- So breaking the RSA is unlikely to imply P = NP.
- But numbers can be factorized efficiently by quantum computers.[b]
- RSA was alleged to have received 10 million US dollars from the government to promote unsecure p and q.[c]

---

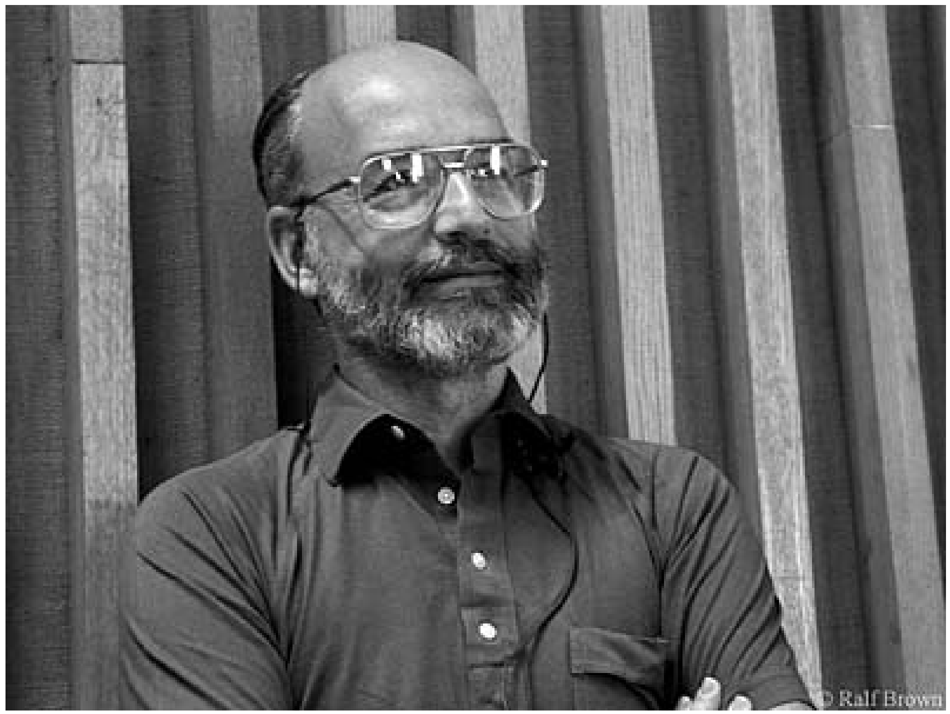
[a] Brassard (1979).
[b] Shor (1994).
[c] Menn (2013).

Adi Shamir, Ron Rivest, and Leonard Adleman

Ron Rivest[a] (1947–)

[a]Turing Award (2002).

Adi Shamir[a] (1952–)



[a]Turing Award (2002).

A Parallel History

- Diffie and Hellman's solution to the secret-key agreement problem led to public-key cryptography.
- In 1973, the RSA public-key cryptosystem was invented in Britain before the Diffie-Hellman secret-key agreement scheme.[a]

[a]Ellis, Cocks, and Williamson of the Communications Electronics Security Group of the British Government Communications Head Quarters (GCHQ).

Is a forged signature the same sort of thing
as a genuine signature,
or is it a different sort of thing?
— Gilbert Ryle (1900–1976),
The Concept of Mind (1949)

"Katherine, I gave him the code.
He verified the code."
"But did you verify him?"
— *The Numbers Station* (2013)

Digital Signatures[a]

- Alice wants to send Bob a *signed* document x.
- The signature must unmistakably identifies the sender.
- Both Alice and Bob have public and private keys

$$e_{\text{Alice}}, e_{\text{Bob}}, d_{\text{Alice}}, d_{\text{Bob}}.$$

- Every cryptosystem guarantees $D(d, E(e, x)) = x$.
- Assume the cryptosystem also satisfies the commutative property

$$E(e, D(d, x)) = D(d, E(e, x)). \tag{16}$$

  – E.g., the RSA system satisfies it as $(x^d)^e = (x^e)^d$.

[a]Diffie & Hellman (1976).

Digital Signatures Based on Public-Key Systems

- Alice signs x as
$$(x, D(d_{\text{Alice}}, x)).$$
- Bob receives (x, y) and verifies the signature by checking
$$E(e_{\text{Alice}}, y) = E(e_{\text{Alice}}, D(d_{\text{Alice}}, x)) = x$$
based on Eq. (16).
- The claim of authenticity is founded on the difficulty of inverting E_{Alice} without knowing the key d_{Alice}.

Blind Signatures[a]

- There are applications where the document author (Alice) and the signer (Bob) are *different* parties.
- Sender privacy: We do not want Bob to see the document.
  - Anonymous electronic voting systems, digital cash schemes, anonymous payments, etc.
- Idea: The document is **blinded** by Alice before it is signed by Bob.
- The resulting blind signature can be publicly verified against the original, unblinded document x as before.

---

[a]Chaum (1983).

Blind Signatures Based on RSA

Blinding by Alice:

1: Pick $r \in Z_n^*$ randomly;

2: Send $x' = xr^e \bmod n$ to Bob; $\{x$ is blinded by $r^e.\}$

- Note that $r \to r^e \bmod n$ is a one-to-one correspondence.
- Hence $r^e \bmod n$ is a random number, too.
- As a result, x' is random and leaks no information.

Blind Signatures Based on RSA (continued)

Signing by Bob with his private decryption key d:

1: Send the blinded signature $s' = (x')^d \bmod n$ to Alice;

Blind Signatures Based on RSA (continued)

The RSA signature of Alice:

1: Alice obtains the signature $s = s' r^{-1} \bmod n$;

- This works because

  $$s \equiv s' r^{-1} \equiv (x')^d r^{-1} \equiv (x r^e)^d r^{-1} \equiv x^d r^{ed-1} \equiv x^d \bmod n$$

  by the properties of the RSA function.

- Note that only Alice knows r.

Blind Signatures Based on RSA (concluded)

- Anyone can verify the document was signed by Bob by checking with Bob's encryption key e the following:

$$s^e \equiv x \bmod n.$$

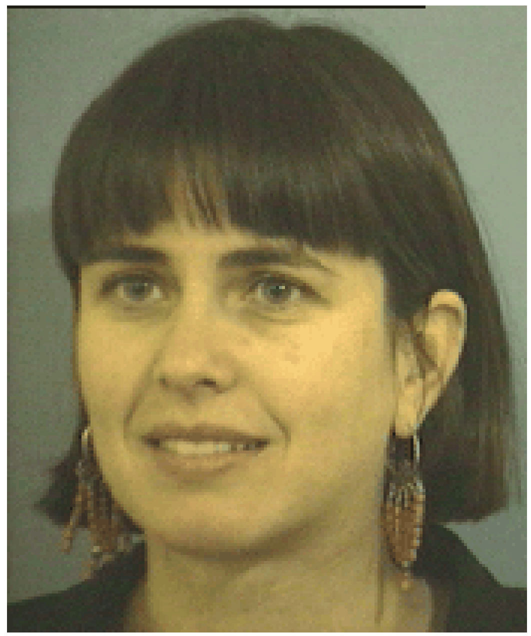
- But Bob does not know s is related to x' (thus Alice).

Probabilistic Encryption[a]

- A deterministic cryptosystem can be broken if the plaintext has a distribution that favors the “easy” cases.
- The ability to forge signatures on even a vanishingly small fraction of strings of some length is a security weakness if those strings were the probable ones!
- A scheme may also “leak” *partial* information.
  - Parity of the plaintext, e.g.
- The first solution to the problems of skewed distribution and partial information was based on the QRA.

[a]Goldwasser & Micali (1982). This paper “laid the framework for modern cryptography” (2013).

Shafi Goldwasser[a] (1958–)



[a]Turing Award (2013).

Silvio Micali[a] (1954–)

[a]Turing Award (2013).

Goldwasser and Micali

A Useful Lemma

Lemma 81 *Let $n = pq$ be a product of two distinct primes. Then a number $y \in Z_n^*$ is a quadratic residue modulo n if and only if $(y \,|\, p) = (y \,|\, q) = 1$.*

- The "only if" part:
  - Let x be a solution to $x^2 = y \bmod pq$.
  - Then $x^2 = y \bmod p$ and $x^2 = y \bmod q$ also hold.
  - Hence y is a quadratic modulo p and a quadratic residue modulo q.

The Proof (concluded)

- The "if" part:
  - Let $a_1^2 = y \bmod p$ and $a_2^2 = y \bmod q$.
  - Solve
    $$\begin{aligned} x &= a_1 \bmod p, \\ x &= a_2 \bmod q, \end{aligned}$$
    for x with the Chinese remainder theorem (p. 486).
  - As $x^2 = y \bmod p$, $x^2 = y \bmod q$, and $\gcd(p, q) = 1$, we must have $x^2 = y \bmod pq$.

The Jacobi Symbol and Quadratic Residuacity Test

- The Legendre symbol can be used as a test for quadratic residuacity by Lemma 68 (p. 554).
- Lemma 81 (p. 680) says this is *not* the case with the Jacobi symbol in general.
- Suppose $n = pq$ is a product of two distinct primes.
- A number $y \in Z_n^*$ with Jacobi symbol $(y \,|\, pq) = 1$ is a quadratic *nonresidue* modulo n when

$$(y \,|\, p) = (y \,|\, q) = -1,$$

  because $(y \,|\, pq) = (y \,|\, p)(y \,|\, q)$.

The Setup

- Bob publishes $n = pq$, a product of two distinct primes, and a quadratic nonresidue y with Jacobi symbol 1.
- Bob keeps secret the factorization of n.
- Alice wants to send bit string $b_1 b_2 \cdots b_k$ to Bob.
- Alice encrypts the bits by choosing a random quadratic residue modulo n if b_i is 1 and a random quadratic nonresidue (with Jacobi symbol 1) otherwise.
- So a sequence of residues and nonresidues are sent.
- Knowing the factorization of n, Bob can efficiently test quadratic residuacity and thus read the message.

The Protocol for Alice

for $i = 1, 2, \ldots, k$ **do**
  Pick $r \in Z_n^*$ randomly;
  if $b_i = 1$ **then**
    Send $r^2 \bmod n$; {Jacobi symbol is 1.}
  else
    Send $r^2 y \bmod n$; {Jacobi symbol is still 1.}
  end if
end for

The Protocol for Bob

for $i = 1, 2, \ldots, k$ **do**

  Receive r;

  if $(r \,|\, p) = 1$ and $(r \,|\, q) = 1$ **then**

    $b_i := 1$;

  else

    $b_i := 0$;

  end if

end for

Semantic Security

- This encryption scheme is probabilistic.
- There are a large number of different encryptions of a given message.
- One is chosen at random by the sender to represent the message.
  - Encryption is a *one-to-many* mapping.
- This scheme is both polynomially secure and **semantically secure**.