# Theory of Computation

homework 3
Due: 11/4/2014

**Problem 1** Show that NL $\neq$ PSPACE.

**Ans:** By Space Hierarchy Theorem (slide p. 234),

$$\text{SPACE}(f(n)) \subsetneq \text{SPACE}(f(n)\log f(n)) \subseteq \text{SPACE}(f^2(n)).$$

By Savitch's Theorem (slide p. 253), for proper function $f(n) \geq \log n$,

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n)).$$

With $f(n) = \log^2 n$, we obtain

$$\text{SPACE}(\log^2 n) \subsetneq \text{SPACE}(\log^4 n).$$

Hence,

$$\text{NL} = \text{NSPACE}(\log n) \subseteq \text{SPACE}(\log^2 n) \subsetneq \text{SPACE}(\log^4 n) \subseteq \text{PSPACE}.$$

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**Problem 2** Show that for $k > 3$, $k$-SAT is NP-complete.

**Ans:** Consider a $k$-SAT expression $\Phi$ with $n$ variables, $m$ clauses and $k$ literals in every clause. First, $k$-SAT $\in$ NP since an NTM can guess a truth assignment for the $n$ variables and then check if it satisfies $\Phi$, all in polynomial time. We proceed to prove $k$-SAT is NP-hard by reducing 3SAT to it. Let $\Phi'$ denote an instance of 3SAT and $c'_j$ be a clause of $\Phi'$, where $j = 1, 2, \ldots, m$. We convert $\Phi'$ to $\Phi$ by adding the new variables $g_1, g_2, \ldots, g_{k-3}$ into each clause as follows. Without loss of generality, set $k = 5$ in our case. We replace each $c'_j = (x \vee y \vee z)$ by

$$c_j = (c'_j \vee g_1 \vee g_2) \wedge (c'_j \vee \neg g_1 \vee g_2) \wedge (c'_j \vee g_1 \vee \neg g_2) \wedge (c'_j \vee \neg g_1 \vee \neg g_2). \quad (1)$$

Note that each clause has 5 literals. Clearly, the above replacement is polynomial-time doable. Next we prove the reduction works.

If $c'_j$ is satisfied by a truth assignment, then $c_j$ is satisfied by the same truth assignment with $g_1$ and $g_2$ arbitrarily set because $g_1$ and $g_2$ do not affect the result. Hence, if $\Phi'$ is satisfiable, then $\Phi$ is, too.

Suppose $c_j$ is satisfied by a truth assignment. As $g_i$ and $\neg g_i$, where $i = 1, 2$, have opposite truth values, then $c'_j$ must be satisfied under the same truth assignment as well, otherwise, one of the clauses in Eq. (1) would be false. Therefore, if $\Phi$ is satisfiable, $\Phi'$ must be satisfiable too. ■