Theory of Computation

Mid-Term Examination on December 17, 2013
Fall Semester, 2013

Problem 1 (25 points)**.** Show that NP = coNP if there exists an NP-complete language that belongs in co-NP.

Proof. Suppose X is NP-complete and $X \in$ coNP. Let a polynomial-time NTM M decide X. For any language $Y \in$ NP, there is a reduction R from Y to X because X is NP-complete. Now, $X \in$ coNP implies $Y \in$ coNP by the closure of reduction; hence

$$\text{NP} \subseteq \text{coNP}.$$

On the other hand, suppose $Y \in$ coNP. Then there is a reduction R' from $\bar{Y}$ to X because $\bar{Y} \in$ NP and X is NP-complete. As a result, for all input strings x,

$$x \in \bar{Y} \text{ iff } R'(x) \in X.$$

This implies $\bar{Y} \in$ coNP by the closure of reduction and the assumption of $X \in$ coNP. Consequently, $Y \in$ NP and

$$\text{coNP} \subseteq \text{NP}.$$

Thus, NP = coNP. ■

Problem 2 (25 points)**.** A cut in an undirected graph $G = (V, E)$ is a partition of the nodes into two nonempty sets S and $V - S$. MAX BISECTION asks if there is a cut of size at least K such that $|S| = |V - S|$. It is known that MAX BISECTION is NP-complete. BISECTION WIDTH asks if there is a bisection of size at most K such that $|S| = |V - S|$. Show that BISECTION WIDTH is NP-complete. You do not need to show it is in NP.

Proof. See pp. 368–369 in the slides. ■

Problem 3 (25 points)**.** Show that 6-COLORING is NP-hard. (6-COLORING asks if a graph can be colored by 6 or fewer colors such that no adjacent nodes have the same color). You do not need to show it is in NP. Recall that 3-COLORING is NP-complete.

Proof. We reduce 3-COLORING to 6-COLORING. Given a graph $G(V, E)$ for 3-COLORING, the reduction outputs a graph $G'(V', E')$ by adding 3 new nodes with edges between each of the 3 nodes and all the other nodes in V. That is, $V' = V \cup \{x_1, x_2, x_3\}$ and $E' = E \cup \{\{x_i, v\} | v \in V', i = 1, 2, 3, x_i \neq v\}$. If $G \in$ 3-COLORING, then $G' \in$ 6-COLORING because 3 or fewer colors for the nodes in V and an additional 3 colors for those in $\{x_1, x_2, x_3\}$ suffice to make a legal coloring. Conversely, consider a legal coloring of G' with 6 or fewer colors. In such a coloring, $\{x_1, x_2, x_3\}$ use up 3 colors, leaving at most 3 colors for the nodes in V. ■

Problem 4 (25 points)**.** We know that 3-SAT is NP-complete. Show that for $n > 3$, n-SAT is also NP-complete. (You don't need to show that is in NP.)

Proof. We reduce 3-SAT to n-SAT as follows. Let ϕ be an instance of 3-SAT. For any clause $(a \vee b \vee c)$, we replace it with $(a \vee b \vee \underbrace{c \vee \cdots \vee c}_{n-2 \text{ times}})$. By repeating this process in all the clauses of ϕ, we get a new boolean expression $\phi' \in n$-SAT. Now, we proceed to show that this is a reduction from 3-SAT to n-SAT as follows:

($\Rightarrow$) From the construction, we see that if a truth assignment satisfies ϕ, then it must satisfy ϕ'.

($\Leftarrow$) Let's notice that if a truth assignment satisfy ϕ', then it must also satisfy ϕ.

From this, we then deduct that ϕ is satisfiable if and only if ϕ' is satisfiable as well, hence 3-SAT is reducible to n-SAT, probing that n-SAT is NP-complete.

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