Exponents and Primitive Roots

- From Fermat's "little" theorem, all exponents divide $p - 1$.
- A primitive root of p is thus a number with exponent $p - 1$.
- Let $R(k)$ denote the total number of residues in $\Phi(p)$ that have exponent k.
- We already knew that $R(k) = 0$ for $k \not| (p - 1)$.
- So
$$\sum_{k \,|\, (p-1)} R(k) = p - 1$$
as every number has an exponent.

Size of $R(k)$

- Any $a \in \Phi(p)$ of exponent k satisfies $x^k = 1 \bmod p$.
- Hence there are at most k residues of exponent k, i.e., $R(k) \leq k$, by Lemma 59 (p. 416).
- Let s be a residue of exponent k.
- $1, s, s^2, \ldots, s^{k-1}$ are distinct modulo p.
  - Otherwise, $s^i = s^j \bmod p$ with $i < j$.
  - Then $s^{j-i} = 1 \bmod p$ with $j - i < k$, a contradiction.
- As all these k distinct numbers satisfy $x^k = 1 \bmod p$, they comprise *all* solutions of $x^k = 1 \bmod p$.

Size of $R(k)$ (continued)

- But do all of them have exponent k (i.e., $R(k) = k$)?
- And if not (i.e., $R(k) < k$), how many of them do?
- Suppose $\ell < k$ and $\ell \notin \Phi(k)$ with $\gcd(\ell, k) = d > 1$.
- Then
$$(s^{\ell})^{k/d} = (s^{k})^{\ell/d} = 1 \bmod p.$$
- Therefore, s^{ℓ} has exponent at most k/d, which is less than k.
- We conclude that
$$R(k) \leq \phi(k).$$

Size of $R(k)$ (concluded)

- Because all $p-1$ residues have an exponent,

$$p-1=\sum_{k|(p-1)} R(k) \leq \sum_{k|(p-1)} \phi(k) = p-1$$

  by Lemma 55 (p. 405).

- Hence

$$R(k)=\begin{cases} \phi(k) & \text{when } k|(p-1) \\ 0 & \text{otherwise} \end{cases}$$

- In particular, $R(p-1)=\phi(p-1)>0$, and p has at least one primitive root.

- This proves one direction of Theorem 51 (p. 393).

A Few Calculations

- Let $p = 13$.
- From p. 413, we know $\phi(p-1) = 4$.
- Hence $R(12) = 4$.
- Indeed, there are 4 primitive roots of p.
- As $\Phi(p-1) = \{1, 5, 7, 11\}$, the primitive roots are g^1, g^5, g^7, g^{11} for any primitive root g.

The Other Direction of Theorem 51 (p. 393)

- We must show p is a prime only if there is a number r (called primitive root) such that
  1. $r^{p-1} = 1 \bmod p$, and
  2. $r^{(p-1)/q} \neq 1 \bmod p$ for all prime divisors q of $p-1$.
- Suppose p is not a prime.
- We proceed to show that no primitive roots exist.
- Suppose $r^{p-1} = 1 \bmod p$ (note $\gcd(r, p) = 1$).
- We will show that the 2nd condition must be violated.

The Proof (continued)

- $r^{\phi(p)} = 1 \bmod p$ by the Fermat-Euler theorem (p. 413).
- Because p is not a prime, $\phi(p) < p - 1$.
- Let k be the smallest integer such that $r^k = 1 \bmod p$.
  - By condition 1, it is easy to show that $k \mid (p-1)$ (p. 416).
- Note that $k \mid \phi(p)$ (p. 416).
- As $k \leq \phi(p)$, $k < p - 1$.
- Let q be a prime divisor of $(p-1)/k > 1$.
- Then $k | (p-1)/q$.

The Proof (concluded)

- Therefore, by virtue of the definition of k,

$$r^{(p-1)/q} = 1 \bmod p.$$

- But this violates the 2nd condition.

Function Problems

- Decision problems are yes/no problems (SAT, TSP (D), etc.).
- **Function problems** require a solution (a satisfying truth assignment, a best TSP tour, etc.).
- Optimization problems are clearly function problems.
- What is the relation between function and decision problems?
- Which one is harder?

Function Problems Cannot Be Easier than Decision Problems

- If we know how to generate a solution, we can solve the corresponding decision problem.
  - If you can find a satisfying truth assignment efficiently, then SAT is in P.
  - If you can find the best TSP tour efficiently, then TSP (D) is in P.
- But decision problems can be as hard as the corresponding function problems.

FSAT

- FSAT is this function problem:
  - Let $\phi(x_1, x_2, \ldots, x_n)$ be a boolean expression.
  - If ϕ is satisfiable, then return a satisfying truth assignment.
  - Otherwise, return "no."
- We next show that if SAT $\in$ P, then FSAT has a polynomial-time algorithm.

An Algorithm for FSAT Using SAT

```
t := ε;
if φ ∈ SAT then
    for i = 1, 2, ..., n do
        if φ[ x_i = true ] ∈ SAT then
            t := t ∪ { x_i = true };
            φ := φ[ x_i = true ];
        else
            t := t ∪ { x_i = false };
            φ := φ[ x_i = false ];
        end if
    end for
    return t;
else
    return "no";
end if
```

Analysis

- There are $\leq n+1$ calls to the algorithm for SAT.[a]
- Shorter boolean expressions than ϕ are used in each call to the algorithm for SAT.
- So if SAT can be solved in polynomial time, so can FSAT.
- Hence SAT and FSAT are equally hard (or easy).

[a] Contributed by Ms. Eva Ou (R93922132) on November 24, 2004.

TSP and TSP (D) Revisited

- We are given n cities $1, 2, \ldots, n$ and integer distances $d_{ij} = d_{ji}$ between any two cities i and j.
- TSP asks for a tour with the shortest total distance.
  - The shortest total distance is at most $\sum_{i,j} d_{ij}$.
    * Recall that the input string contains $d_{11}, \ldots, d_{nn}$.
    * Thus the shortest total distance is at most $2^{|\,x\,|}$, where x is the input.
- TSP (D) asks if there is a tour with a total distance at most B.
- We next show that if TSP (D) $\in$ P, then TSP has a polynomial-time algorithm.

An Algorithm for TSP Using TSP (D)

```
Perform a binary search over interval [0, 2^{|x|}] by calling
TSP (D) to obtain the shortest distance, C;
for i, j = 1, 2, ..., n do
    Call TSP (D) with B = C and d_{ij} = C + 1;
    if "no" then
        Restore d_{ij} to old value; {Edge [i, j] is critical.}
    end if
end for
return the tour with edges whose d_{ij} ≤ C;
```

Analysis

- An edge that is not on *any* optimal tour will be eliminated, with its d_{ij} set to $C + 1$.
- An edge which is not on *all remaining* optimal tours will also be eliminated.
- So the algorithm ends with n edges which are not eliminated (why?).
- There are $O(| x | + n^2)$ calls to the algorithm for TSP (D).
- So if TSP (D) can be solved in polynomial time, so can TSP.
- Hence TSP (D) and TSP are equally hard (or easy).

Function Problems Are Not Harder than Decision Problems If P = NP

Theorem 60 *Suppose that $P = NP$. Then, for every NP language L there exists a polynomial-time TM B that on input $x \in L$ outputs a certificate for x.*

- We are looking for a certificate in the sense of Proposition 34 (p. 267).
- That is, a certificate y for every $x \in L$ such that

$$(x, y) \in R,$$

  where R is a polynomially decidable and polynomially balanced relation.

The Proof (concluded)

- Recall the algorithm for FSAT on p. 428.
- The reduction of Cook's Theorem L to SAT is a Levin reduction (p. 271).
- So there is a polynomial-time computable function R such that $x \in L$ iff $R(x) \in$ SAT.
- In fact, more is true: R maps a satisfying assignment of $R(x)$ into a certificate for x.
- Therefore, we can use the algorithm for FSAT to come up with an assignment for $R(x)$ and then map it back into a certificate for x.

What If NP = coNP?[a]

- Can you say similar things?

[a]Contributed by Mr. Ren-Shuo Liu (`D98922016`) on October 27, 2009.

Randomized Computation

I know that half my advertising works,
I just don't know which half.
— John Wanamaker

I know that half my advertising is
a waste of money,
I just don't know which half!
— McGraw-Hill ad.

Randomized Algorithms[a]

- Randomized algorithms flip unbiased coins.
- There are important problems for which there are no known efficient *deterministic* algorithms but for which very efficient randomized algorithms exist.
  - Extraction of square roots, for instance.
- There are problems where randomization is *necessary*.
  - Secure protocols.
- Randomized version can be more efficient.
  - Parallel algorithm for maximal independent set.

[a] Rabin (1976); Solovay and Strassen (1977).

"Four Most Important Randomized Algorithms"[a]

1. Primality testing.[b]
2. Graph connectivity using random walks.[c]
3. Polynomial identity testing.[d]
4. Algorithms for approximate counting.[e]

---

[a] Trevisan (2006).
[b] Rabin (1976); Solovay and Strassen (1977).
[c] Aleliunas, Karp, Lipton, Lovász, and Rackoff (1979).
[d] Schwartz (1980); Zippel (1979).
[e] Sinclair and Jerrum (1989).

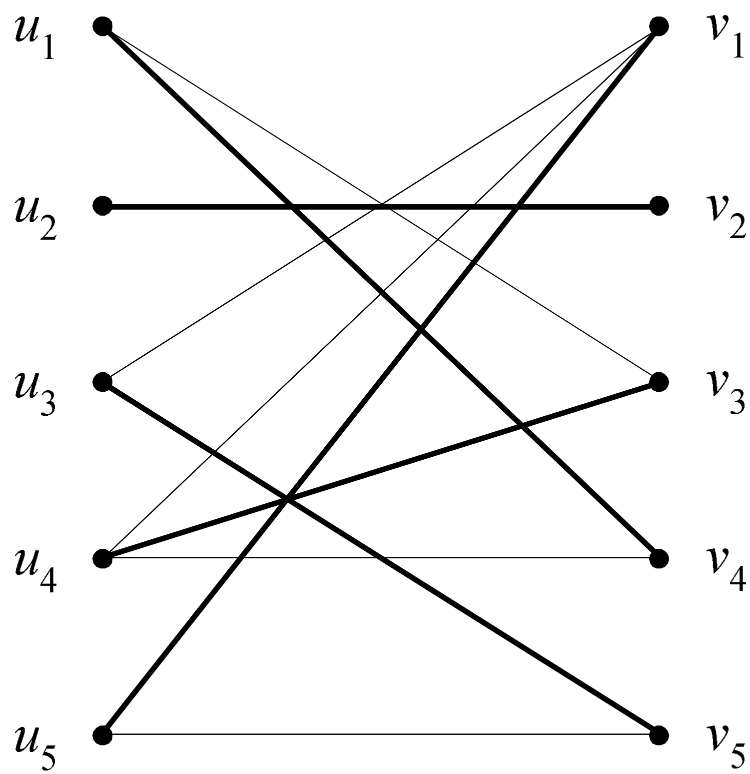
Bipartite Perfect Matching

- We are given a **bipartite graph** $G = (U, V, E)$.
  - $U = \{u_1, u_2, \ldots, u_n\}$.
  - $V = \{v_1, v_2, \ldots, v_n\}$.
  - $E \subseteq U \times V$.
- We are asked if there is a **perfect matching**.
  - A permutation π of $\{1, 2, \ldots, n\}$ such that

    $$(u_i, v_{\pi(i)}) \in E$$

    for all $u_i \in U$.

A Perfect Matching



Symbolic Determinants

- We are given a bipartite graph G.
- Construct the $n \times n$ matrix A^G whose (i, j)th entry A_{ij}^G is a variable x_{ij} if $(u_i, v_j) \in E$ and zero otherwise.

Symbolic Determinants (concluded)

- The **determinant** of A^G is

$$\det(A^G) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} A^G_{i,\pi(i)}. \tag{5}$$

  - π ranges over all permutations of n elements.
  - $\operatorname{sgn}(\pi)$ is 1 if π is the product of an even number of transpositions and -1 otherwise.
  - Equivalently, $\operatorname{sgn}(\pi) = 1$ if the number of (i, j)s such that $i < j$ and $\pi(i) > \pi(j)$ is even.[a]

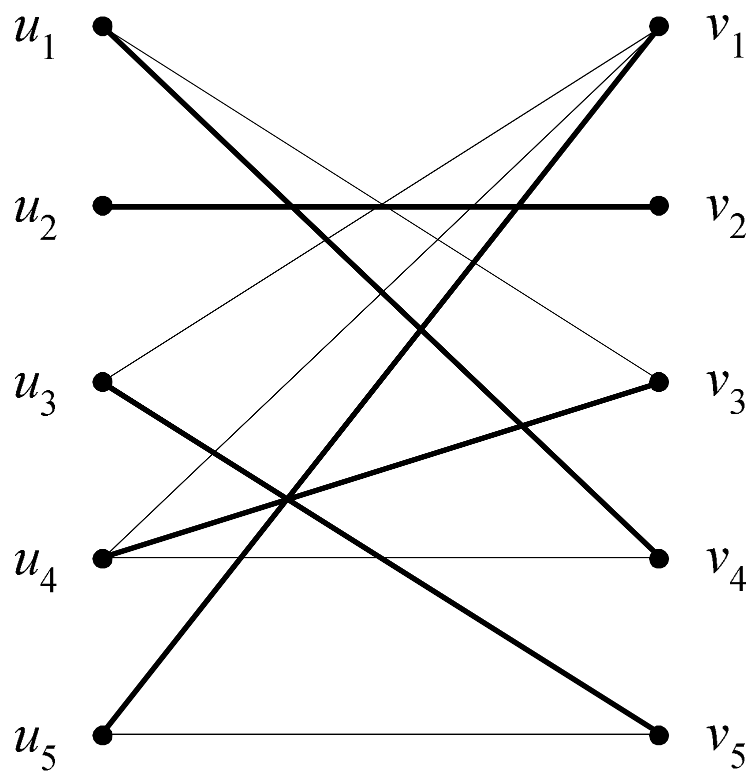
[a] Contributed by Mr. Hwan-Jeu Yu (`D95922028`) on May 1, 2008.

Determinant and Bipartite Perfect Matching

- In $\sum_\pi \text{sgn}(\pi) \prod_{i=1}^n A^G_{i,\pi(i)}$, note the following:
  - Each summand corresponds to a possible perfect matching π.
  - As all variables appear only *once*, all of these summands are different monomials and will not cancel.
- It is essentially an exhaustive enumeration.

Proposition 61 (Edmonds (1967)) *G has a perfect matching if and only if* $\det(A^G)$ *is not identically zero.*

A Perfect Matching in a Bipartite Graph



The Perfect Matching in the Determinant

- The matrix is

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & \boxed{x_{14}} & 0 \\ 0 & \boxed{x_{22}} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & \boxed{x_{35}} \\ x_{41} & 0 & \boxed{x_{43}} & x_{44} & 0 \\ \boxed{x_{51}} & 0 & 0 & 0 & x_{55} \end{bmatrix}.$$

- $\det(A^G) = -x_{14}x_{22}x_{35}x_{43}x_{51} + x_{13}x_{22}x_{35}x_{44}x_{51} + x_{14}x_{22}x_{31}x_{43}x_{55} - x_{13}x_{22}x_{31}x_{44}x_{55}$, each denoting a perfect matching.

How To Test If a Polynomial Is Identically Zero?

- $\det(A^G)$ is a polynomial in n^2 variables.
- There are exponentially many terms in $\det(A^G)$.
- Expanding the determinant polynomial is not feasible.
  - Too many terms.
- Observation: If $\det(A^G)$ is *identically zero*, then it remains zero if we substitute *arbitrary* integers for the variables $x_{11}, \ldots, x_{nn}$.
- What is the likelihood of obtaining a zero when $\det(A^G)$ is *not* identically zero?

Number of Roots of a Polynomial

Lemma 62 (Schwartz (1980)) *Let* $p(x_1, x_2, \ldots, x_m) \not\equiv 0$ *be a polynomial in* m *variables each of degree at most* d. *Let* $M \in \mathbb{Z}^+$. *Then the number of* m*-tuples*

$$(x_1, x_2, \ldots, x_m) \in \{0, 1, \ldots, M-1\}^m$$

such that $p(x_1, x_2, \ldots, x_m) = 0$ *is*

$$\leq mdM^{m-1}.$$

- By induction on m (consult the textbook).

Density Attack

- The density of roots in the domain is at most
$$\frac{mdM^{m-1}}{M^m} = \frac{md}{M}. \tag{6}$$
- So suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
- Then a random
$$(x_1, x_2, \ldots, x_m) \in \{ 0, 1, \ldots, M-1 \}^m$$
has a probability of $\leq md/M$ of being a root of p.
- Note that M is under our control.

Density Attack (concluded)

Here is a sampling algorithm to test if $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.

```
Choose i_1, ..., i_m from {0, 1, ..., M − 1} randomly;
if p(i_1, i_2, ..., i_m) ≠ 0 then
    return "p is not identically zero";
else
    return "p is probably identically zero";
end if
```

A Randomized Bipartite Perfect Matching Algorithm[a]

We now return to the original problem of bipartite perfect matching.

Choose n^2 integers $i_{11}, \ldots, i_{nn}$ from $\{0, 1, \ldots, 2n^2 - 1\}$ randomly;
Calculate $\det(A^G(i_{11}, \ldots, i_{nn}))$ by Gaussian elimination;
if $\det(A^G(i_{11}, \ldots, i_{nn})) \neq 0$ **then**
    return "G has a perfect matching";
else
    return "G has no perfect matchings";
end if

---

[a]Lovász (1979). According to Paul Erdős, Lovász wrote his first significant paper "at the ripe old age of 17."

Analysis

- If G has no perfect matchings, the algorithm will always be correct.
- Suppose G has a perfect matching.
  - The algorithm will answer incorrectly with probability at most $n^2 d/(2n^2) = 0.5$ with $d = 1$ in Eq. (6) on p. 449.
  - Run the algorithm *independently* k times and output "G has no perfect matchings" if they all say no.
  - The error probability is now reduced to at most 2^{-k}.
- Is there an $(i_{11}, \ldots, i_{nn})$ that will always give correct answers for all bipartite graphs of $2n$ nodes?[a]

[a]Thanks to a lively class discussion on November 24, 2004.

Analysis (concluded)[a]

- Note that we are calculating

$$\text{prob}[\,\text{algorithm answers "no"} \,|\, G \text{ has no perfect matchings}\,],$$
$$\text{prob}[\,\text{algorithm answers "yes"} \,|\, G \text{ has a perfect matching}\,].$$

- We are *not* calculating

$$\text{prob}[\,G \text{ has no perfect matchings} \,|\, \text{algorithm answers "no"}\,],$$
$$\text{prob}[\,G \text{ has a perfect matching} \,|\, \text{algorithm answers "yes"}\,].$$

[a] Thanks to a lively class discussion on May 1, 2008.

But How Large Can $\det(A^G(i_{11}, \ldots, i_{nn}))$ Be?

- It is at most
$$n!\left(2n^2\right)^n.$$
- Stirling's formula says $n! \sim \sqrt{2\pi n}\,(n/e)^n$.
- Hence
$$\log_2 \det(A^G(i_{11}, \ldots, i_{nn})) = O(n \log_2 n)$$
bits are sufficient for representing the determinant.
- We skip the details about how to make sure that all intermediate results are of polynomial sizes.

Lószló Lovász (1948–)

Perfect Matching for General Graphs

- Page 440 is about bipartite perfect matching
- Now we are given a graph $G = (V, E)$.
  - $V = \{v_1, v_2, \ldots, v_{2n}\}$.
- We are asked if there is a perfect matching.
  - A permutation π of $\{1, 2, \ldots, 2n\}$ such that
$$(v_i, v_{\pi(i)}) \in E$$
    for all $v_i \in V$.

The Tutte Matrix[a]

- Given a graph $G = (V, E)$, construct the $2n \times 2n$ **Tutte matrix** T^G such that

$$T^G_{ij} = \begin{cases} x_{ij} & \text{if } (v_i, v_j) \in E \text{ and } i < j, \\ -x_{ij} & \text{if } (v_i, v_j) \in E \text{ and } i > j, \\ 0 & \text{othersie.} \end{cases}$$

- The Tutte matrix is a skew-symmetric symbolic matrix.
- Similar to Proposition 61 (p. 444):

Proposition 63 *G has a perfect matching if and only if* $\det(T^G)$ *is not identically zero.*

---

[a]William Thomas Tutte (1917–2002).

William Thomas Tutte (1917–2002)