Problem 1 (25 points). Show that if $\text{NP} \subseteq \text{NSPACE}(n^2)$, then $\text{NP} \neq \text{PSPACE}$.

Proof. By Savitch’s theorem, $\text{NSPACE}(n^2) \subseteq \text{SPACE}(n^4)$. By the space-hierarchy theorem, $\text{SPACE}(n^4) \subset \text{PSPACE}$. Hence $\text{NP} \subseteq \text{NSPACE}(n^2)$ implies $\text{NP} \subseteq \text{NSPACE}(n^2) \subseteq \text{SPACE}(n^4) \subset \text{PSPACE}$.

Problem 2 (25 points). Let the mix Hamiltonian path problem ask whether, given two undirected graphs, exactly one of them has a Hamiltonian path. Prove or disprove that mix Hamiltonian path is NP-hard.

Proof. We present a logarithmic-space reduction from Hamiltonian path to mix Hamiltonian path, thus establishing the NP-hardness of mix Hamiltonian path. Let $G_0$ be a fixed undirected graph without a Hamiltonian path. Given an undirected graph $G$, the reduction outputs $G$ and $G_0$. Clearly, $G$ has a Hamiltonian path if and only if exactly one of $G$ and $G_0$ does.

Problem 3 (25 points). It is known that EXP-hard languages exist. Can every NP-complete language be reduced to an EXP-hard language? Briefly justify your answer.

Proof. Clearly, $\text{NP} \subseteq \text{EXP}$. By definition, all languages in $\text{EXP}$, including the NP-complete ones, can be reduced to an EXP-hard language.

Problem 4 (25 points). Show that if both $L$ and $\overline{L}$ are recursively enumerable languages, then $L$ is recursive.

Proof. Suppose that $L$ and $\overline{L}$ are accepted by Turing machines $M$ and $\overline{M}$, respectively. Then $L$ is decided by Turing machine $M'$, defined as follows. On input $x$, $M'$ simulates on two different strings both $M$ and $\overline{M}$ in an
interleaved fashion. That is, it simulates a step of $M$ on one string, then a step of $\bar{M}$ on the other, then again another step of $M$ on the first, and so on. Since $M$ accepts $L$, $\bar{M}$ accepts its complement and $x$ must be in one of the two, it follows that one of the two machines will halt and accept. If $M$ accepts, then $M'$ halts on state “yes.” If $\bar{M}$ accepts, then $M'$ halts on “no.”