Problem 1 (15 points). Please answer the following questions.

1. [5 points] Is \( \text{EXP} \subseteq \text{NEXP} \) true?

2. [5 points] Does there exist a language in NP decidable in time linear in the input length?

3. [5 points] In 1975, Richard Ladner proved the famous theorem that if \( \text{NP} \neq \text{P} \), then there is a language in NP that is neither in P nor NP-complete. Is the converse of the theorem true? That is, if some language in NP is neither in P nor NP-complete, can we conclude \( \text{NP} \neq \text{P} \)?

Solution. The questions are answered below.

1. Yes because a deterministic Turing machine is by definition a degenerate nondeterministic Turing machine.

2. Yes, the empty language belongs to \( \text{NP} \cap \text{TIME}(n) \).

3. Yes.

Problem 2 (20 points). Show that there exists a language \( L \subseteq \{0, 1\}^* \) that belongs neither to RE nor to coRE.

Solution. Since there are countably many Turing machines, \( \text{RE} \cup \text{coRE} \) is countable. Hence we only need to show that there are uncountably many subsets of \( \{0, 1\}^* \) to complete the proof.

Assume for contradiction that there are only countably many subsets of \( \{0, 1\}^* \) and use \( L_0, L_1, \ldots \) to denote these subsets. Write \( \{0, 1\}^* = \{ x_k \mid k \in \mathbb{N} \} \) where \( x_i \neq x_j \) for \( i \neq j \). Then the subset \( \hat{L} = \{ x_k \mid x_k \notin L_k, k \in \mathbb{N} \} \) of \( \{0, 1\}^* \) must equal \( L_t \) for some \( t \in \mathbb{N} \). But \( x_t \in \hat{L} \) and \( x_t \notin \hat{L} \) imply each other, which is absurd.
Problem 3 (20 points). Show that there exists a language in NPSPACE that is not decidable in time cubic in the input length.

Solution. The time hierarchy theorem implies \( \text{TIME}(n^3) \subseteq \text{P} \). This and the trivial fact that \( \text{P} \subseteq \text{NPSPACE} \) complete the proof.

Problem 4 (20 points). Does there exist a non-recursive language in NP?

Solution. No, every language in NP can be decided in at most exponential time.

Problem 5 (20 points). Show that it is NP-hard to determine whether a Boolean expression in conjunctive normal form has at least two satisfying assignments. (Hint: Consider adding a clause \( C \) to a Boolean expression \( F \) in conjunctive normal form where the variables in \( C \) do not appear in \( F \).)

Solution. We describe a logspace reduction from SAT to the problem in question. On input a Boolean expression \( F \) in conjunctive normal form, the reduction outputs \( F' = F \land (x \lor y \lor z) \) where \( x, y, z \) are variables that do not appear in \( F \). Now if \( F \) is satisfiable, then \( F' \) has at least 7 satisfying assignments because \( (x \lor y \lor z) \) is satisfied by seven assignments to \( x, y \) and \( z \). Conversely, if \( F' \) has at least two satisfying assignments, it is clear that \( F \) must have at least one.

Problem 6 (20 points). Prove that \( \text{NSPACE}(\log^2 n) \subseteq \text{TIME}(2^{(\log^4 n)}) \).

Solution. A nondeterministic Turing machine \( M \) with space complexity \( O(\log^2 n) \) goes through no more than \( c^{\log^2 |x|} \) configurations on any input \( x \), for some \( c > 1 \) depending on \( M \). Consider a directed graph \( G = (V, E) \) where \( V \) consists of those configurations of \( M \) on \( x \) and \( E \) consists of those pairs \((c_1, c_2)\) of configurations such that \( c_1 \) yields \( c_2 \). It takes \( \text{poly}(c^{\log^2 |x|}) \) time to enumerate the vertices and edges of \( G \), and \( \text{poly}(c^{\log^2 |x|}) \) time by breadth-first or depth-first search to determine whether the initial configuration of \( M \) on \( x \) yields an accepting configuration. Hence \( \text{NSPACE}(\log^2 n) \subseteq \text{TIME}(O(1)^{\log^2 n}) \).

Another solution. Savitch’s theorem implies \( \text{NSPACE}(\log^2 n) \subseteq \text{SPACE}(\log^4 n) \). Hence \( \text{NSPACE}(\log^2 n) \subseteq \text{SPACE}(\log^4 n) \subseteq \text{TIME}(O(1)^{\log^4 n}) \).