The Minimum-Area Spanning Tree Problem

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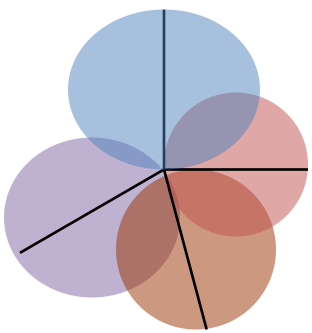
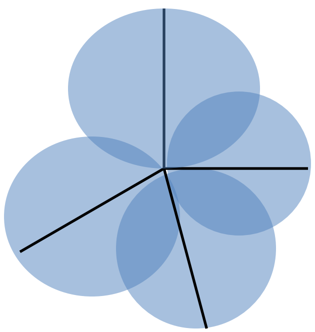
Introduction

- Minimum-Area Spanning Tree(MAST):

  Given a set P of n points in the plane, find a spanning tree of P of minimum area.
- Area:

  union of the n-1 disks whose diameters are edges in T





= Area

- Main result:

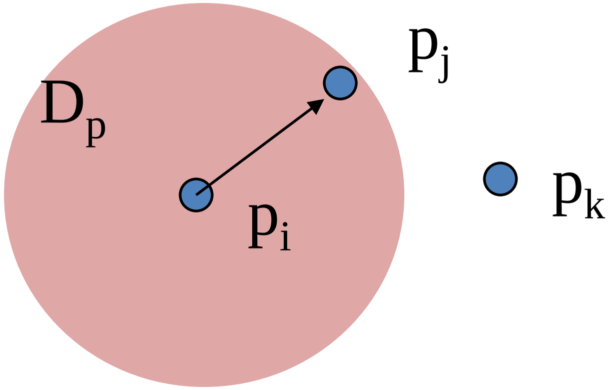
  minimum spanning tree of P is a constant-factor approximation for MAST

Extension problems

- Power assignment problem

  Before Minimum-Area Range Assignment(MARA)

- P: a set of n points(transmitters-receivers)
- Goal:
  - The resulting directed communication graph is strongly connected

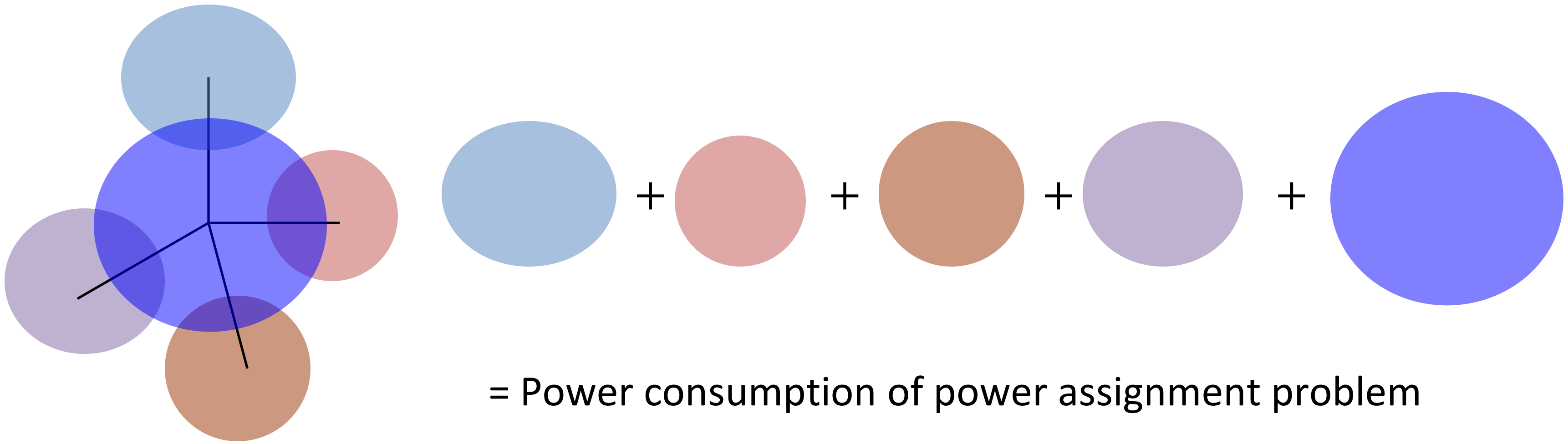


Radius of Dp:

the transmission range assign to Pi

Extension problems

- Power assignment problem
- Goal:
  - The total power consumption is minimal



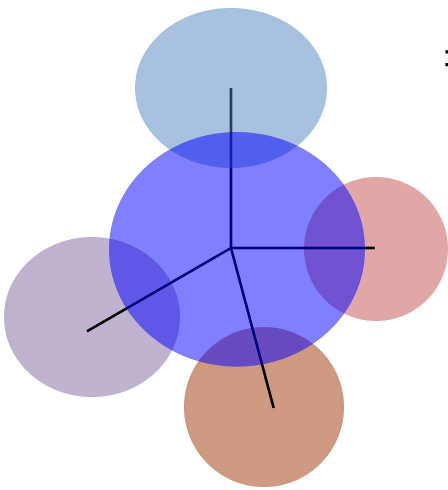
- Result: NP-hard, 2-approximation based on MST

Extension problems

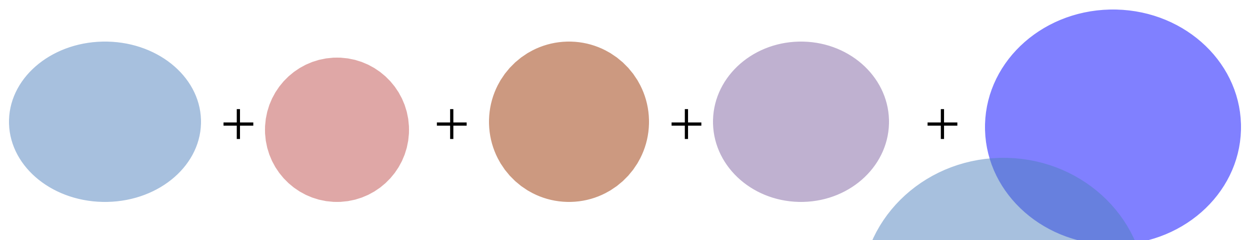
- Minimum-Area Range Assignment(MARA)

  power assignment problem in radio networks

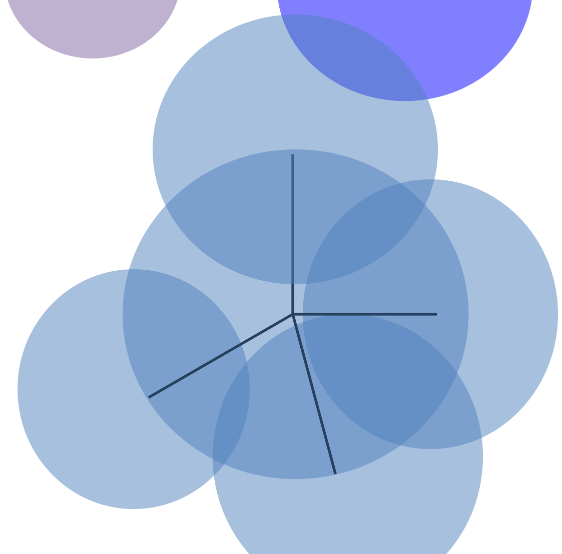
- Goal:
  - Minimize the union of the disks Dp_1,..Dp_n(total coverage area)
  - Prevent from foreign receiver



= Power consumption of power assignment problem

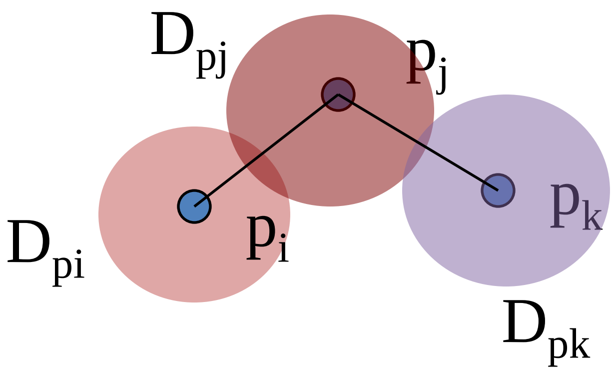


coverage area of MARA =



Extension problems

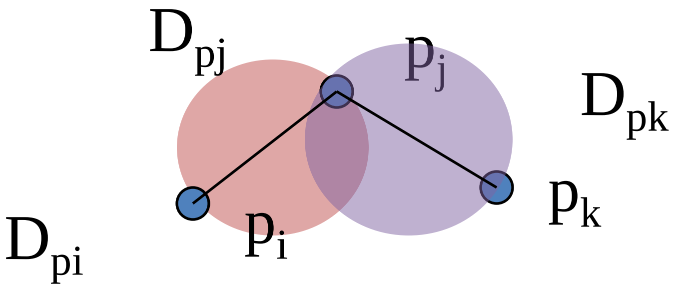
- Minimum-Area Connected Disk Graph(MACDG)
- Goal:
  - the resulting disk graph is connected.



  - Minimize the union of the disks Dp_1,..Dp_n

Extension problems

- Minimum area tour (MAT)
  - Variant of traveling salesman problem
- Goal:
  - Minimize a tour of P of minimum area

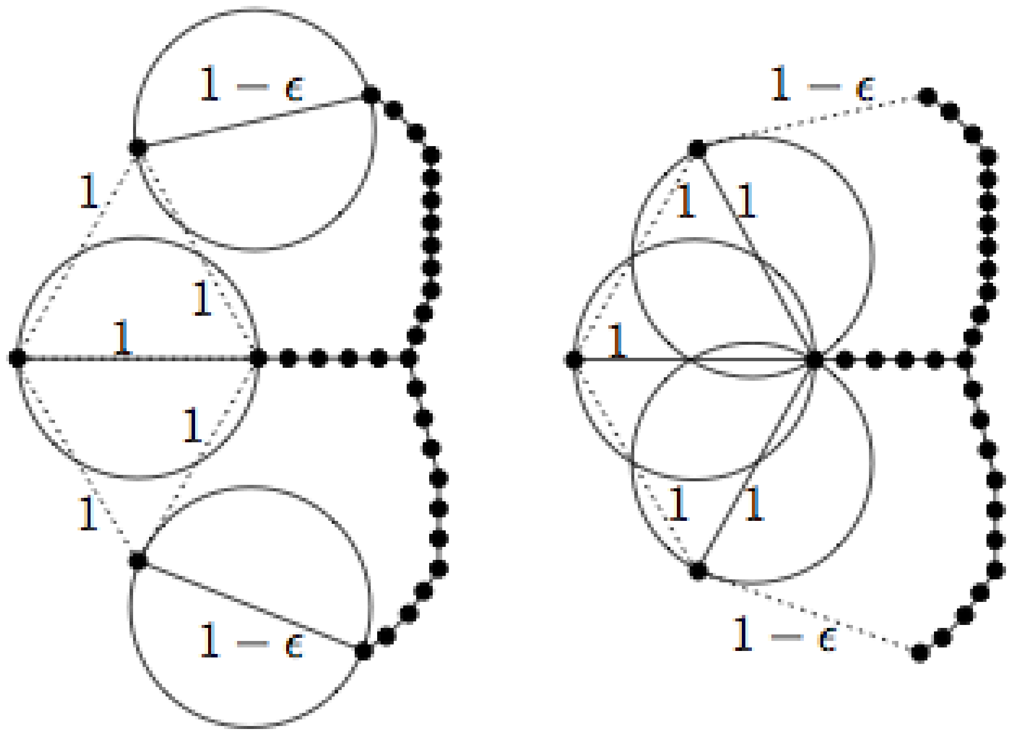


- constant-factor approximation based on relaxed triangle inequality

MST v.s. MAST

- MST: Minimum Spanning Tree
- MAST: Minimum Area Spanning Tree
- We'll prove that MST is a c-approximation for MAST

MST v.s. MAST (Cont.)



(a) (b)

Some Definitions

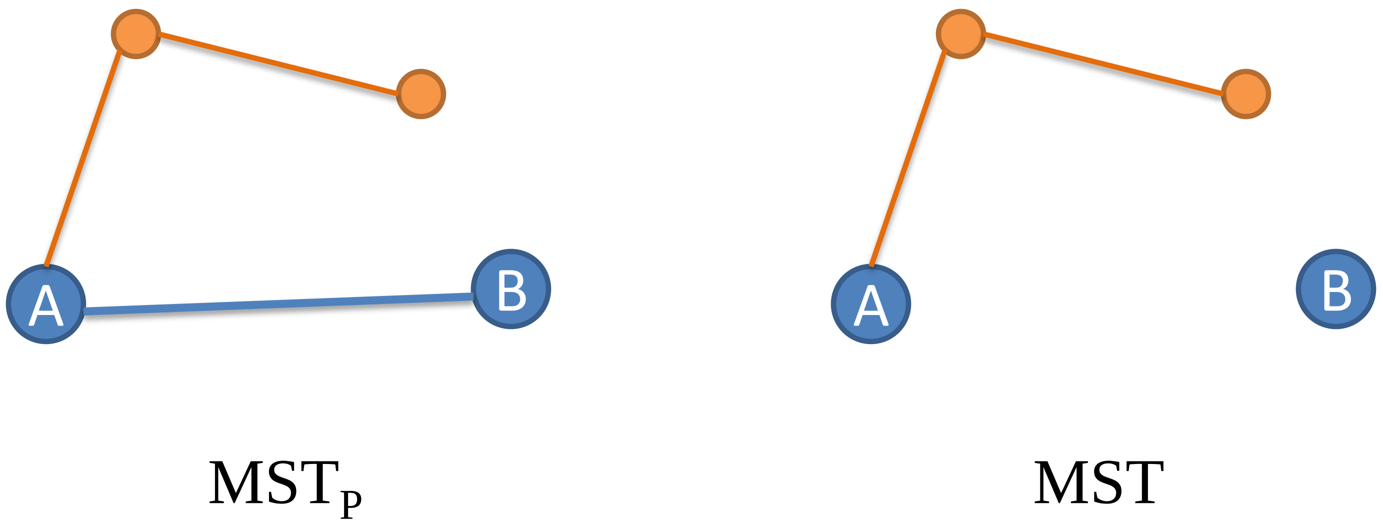
- Let T be any spanning tree of P
- For an edge e in T
  - D(e) is the disk whose diameter is e
  - $D(T) = \{D(e) \mid e \text{ is an edge in } T\}$
  - $U_T = \bigcup_{e \in T} D(e)$
  - $\sigma_T = \sum_{e \in T} \text{area}(D(e))$
- We'll prove that MST is a c-approximation for MAST
  - $\text{area}(U_{MST}) = O(\text{area}(U_{OPT}))$

Claim 1

- Let MST_p be a MST for $P \cup \{p\}$
- There is no edge (a, b) in MST_p, such that (a, b) is not in MST and both a and b are points of P

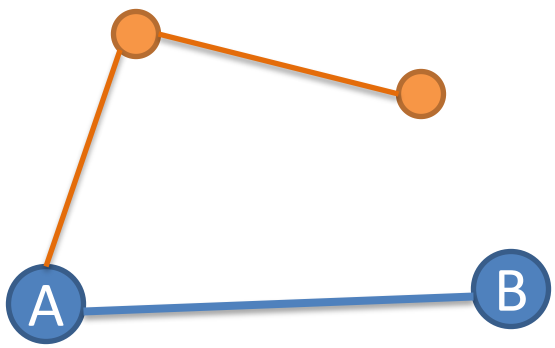
Claim 1 Proof

- Assume there is an edge (a, b) in MST_p but not in MST

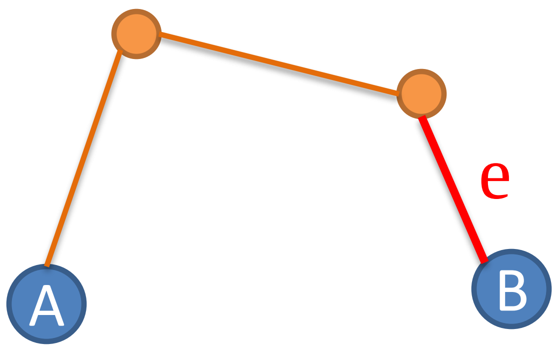


Claim 1 Proof (Cont.)

- Consider the path in MST between a and b
- At least one of the edges along this path is not in MST_p (edge e)



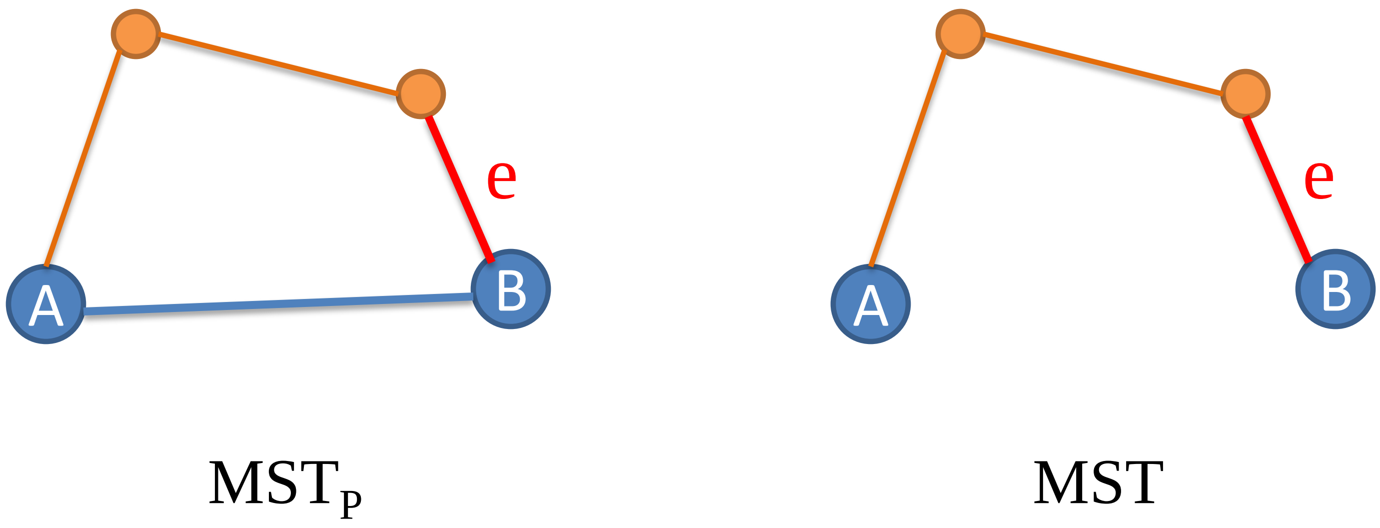
MST_P



MST

Claim 1 Proof (Cont.)

- |e| < |(a, b)|
- We can replace (a, b) in MST_p by e, without increasing the total weight



An Corollary of Claim 1

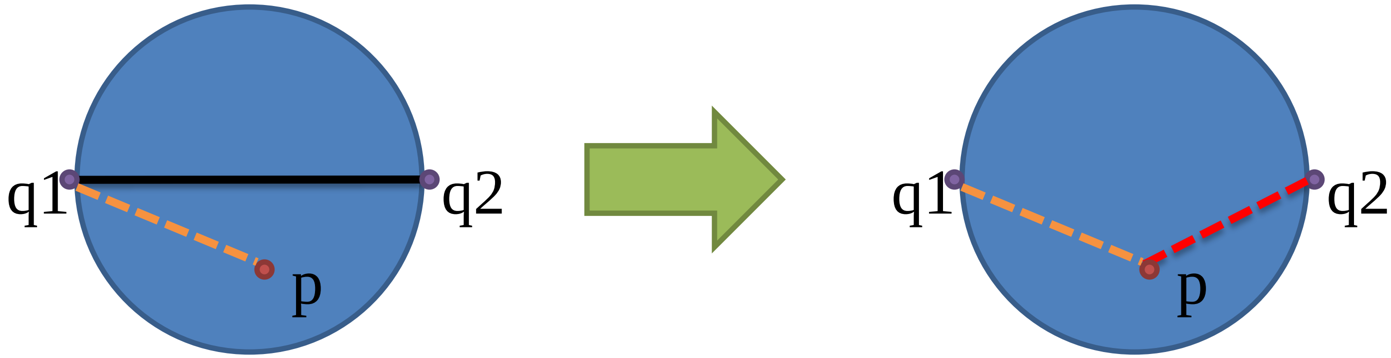
- If e is an edge in MST_p but not in MST, then one of e's endpoints is p

Lemma 1

- $\sigma_{MST} \leq 5$ area(U_{MST})
- At first, we need to prove that p belongs at most 5 of the disks in D(MST)

Lemma 1 Proof

- Let D(q1, q2) be a disk in D(MST)
- If p ∈ D(q1, q2), the edge(q1, q2) is not in MST_p
  - If it is, we can replace (q1, q2) by (p, q1) or (p, q2) to decrease total weight of MST_p



Lemma 1 Proof (Cont.)

- By the corollary of Claim 1
  - If e is an edge in MST_p but not in MST, then one of e's endpoints is p
- Each disk D ∈ D(MST) such that p ∈ D, induces a distinct a distinct edge in MST_p
- The degree of p is at most 6
  - This is true for any vertex of any Euclidean MST

Lemma 1 Proof (Cont.)

- So there can be at most 5 disks covering p
- Then we prove that $\sigma_{MST} \leq 5\ \text{area}(U_{MST})$

ST Construction

- Let OPT be an optimal spanning tree of P, i.e., a solution to MAST.
- We use OPT to construct another spanning tree ST of P

ST Construction (Cont.)

- Initially ST is empty
- In the i'th iteration, let e_i be the longest edge in OPT and there is no path in ST between its endpoints
- Draw two concentric circles C_i and C_i^3 around the mid point of e_i
  - The diameter of C_i is $|e_i|$
  - The diameter of C_i^3 is $3|e_i|$

ST Construction (Cont.)

- Apply Kruskal's MST algorithm with the modification to the points of P lying in C_i^3
  - The edge can't be already in ST
  - The edge can't create cycle in ST
- S_i is the edge set return by Kruskal algorithm in i'th iteration, and we add S_i to ST

ST Construction Examples

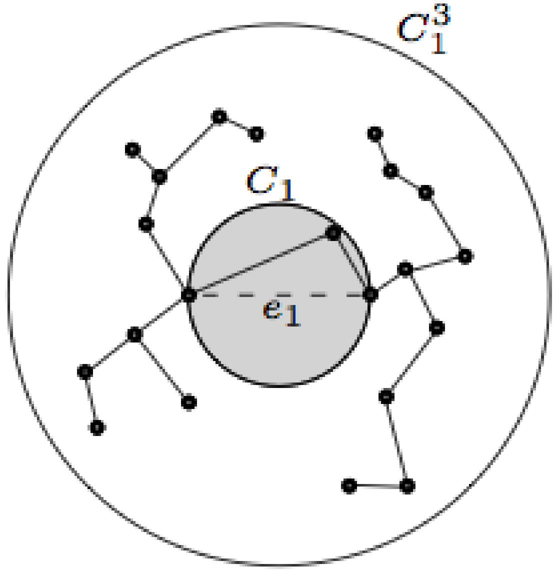
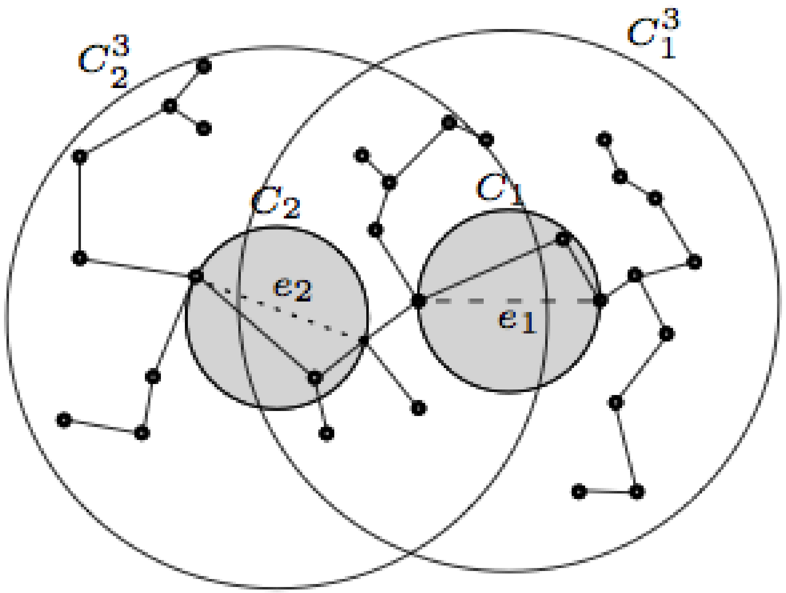


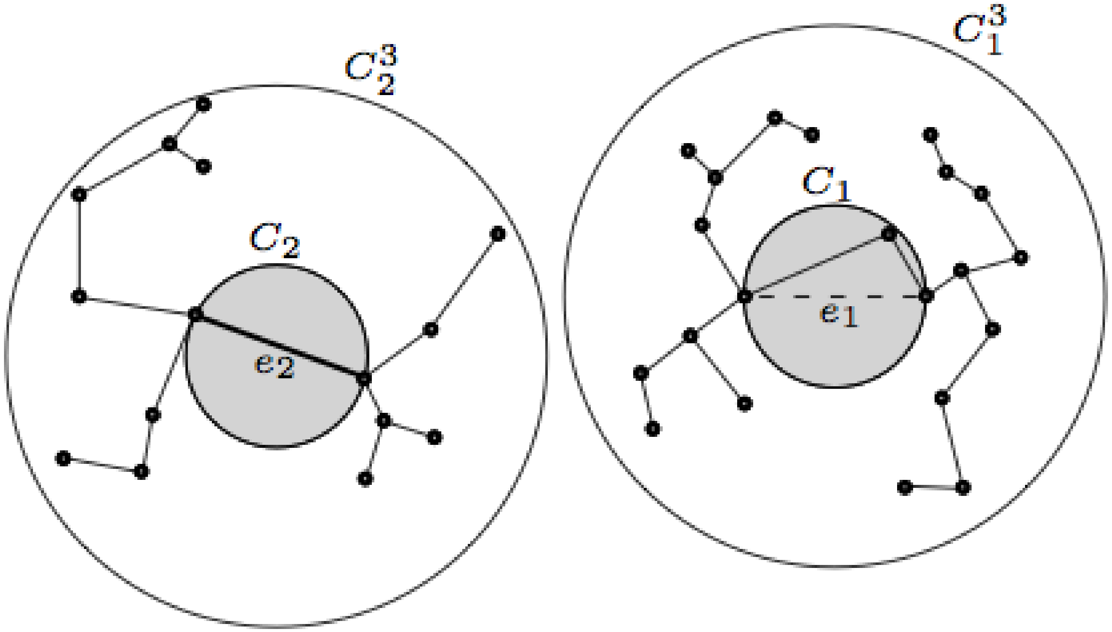
Fig. 3. ST after choosing e_1.

ST Construction Examples



(a)

ST Construction Examples



(b)

Claim 2

- For each *i*, S_i is a subset of the edge set of the minimum spanning tree MST_i that is obtained by applying Kruskal's algorithm, without the modification above, to the points in C_i^3

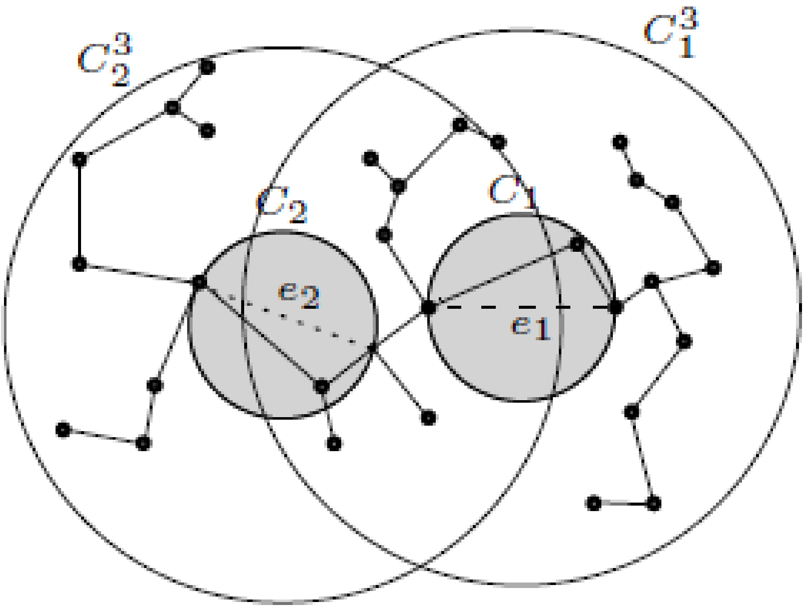
Claim 3: *ST* is a spanning tree of *P*

- Proof

  - there are no cycles in ST.
  - ST is connected, since otherwise there still exists an edge in opt that forces another iteration of the construction algorithm.

Claim 4

- *For any pair of disks* C_i, C_j *in* $\mathcal{C}$, $i \neq j$, *it holds that* $C_i \cap C_j = \emptyset$.



Claim 5

- $\sigma_{\mathrm{ST}} = O(\mathrm{area}(\bigcup_{\mathrm{OPT}}))$

(by lemma 1)

$$\sigma_{S_i} \leq^1 \sigma_{\mathrm{MST}_i} \leq^2 5\,\mathrm{area}(\bigcup_{\mathrm{MST}_i}) =^3 O(\mathrm{area}(C_i^3)) =^4 O(\mathrm{area}(C_i))$$

(by claim 2)

$$\sigma_{\mathrm{ST}} = \Sigma_i \sigma_{S_i} = \Sigma_i O(\mathrm{area}(C_i)) = O(\mathrm{area}(\bigcup_{\mathcal{C}}))$$

(by claim 4)

$\sigma_{\mathrm{ST}} = O(\mathrm{area}(\bigcup_{\mathrm{OPT}}))$ (C is a subset of D(OPT))

Theorem 1

- MST is a constant-factor approximation for MAST

$$\text{area}\Big(\bigcup_{\text{MST}}\Big) \leq^1 \sigma_{\text{MST}} \leq^2 \sigma_{\text{ST}} \leq^3 c \cdot \text{area}\Big(\bigcup_{\text{OPT}}\Big)$$

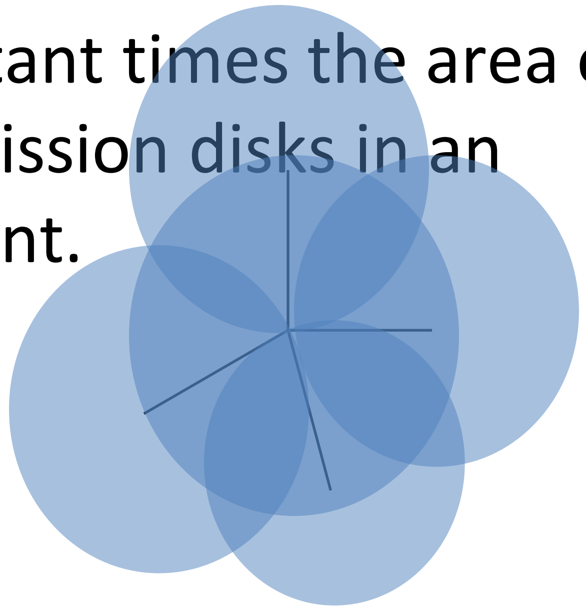
(by claim 5)

Constant-Factor Approximation for Minimum-Area Range Assignment

- Let $p_i \in \boldsymbol{P}$ and r_i is the length of the longest edge in MST that is connected to p_i.
- RA = { $D_{p_1},\ldots,D_{p_n}$}, where D_{pi} is the disk of radius r_i, centered at p_i.
- Let OPT^R denote an optimal range assignment.

MARA problem

- Problem definition
  - The corresponding(directed) communication graph is strongly connected.
  - The area of the union of the disks in RA is bounded by some constant times the area of the union of the transmission disks in an optimal range assignment.



Claim 6: area(U_{RA}) ≤ 9·area(U_{MST})

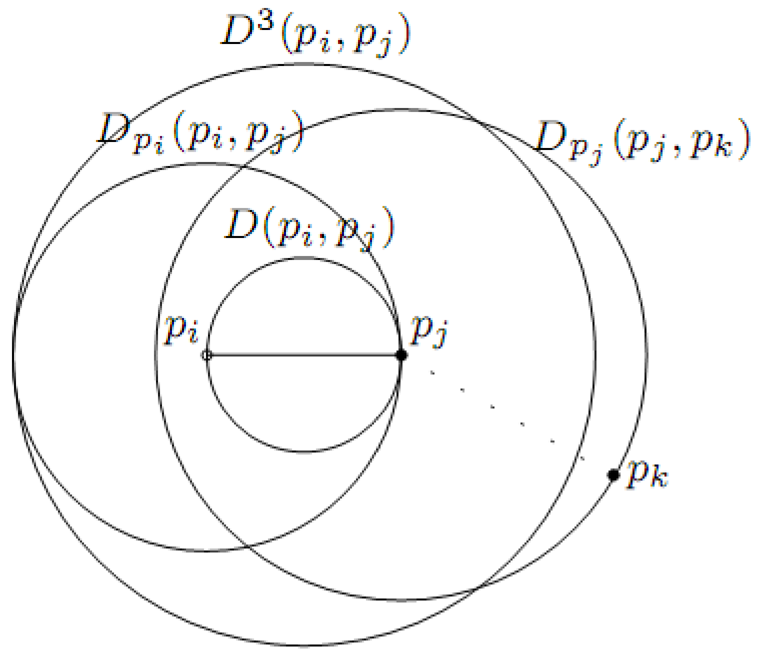
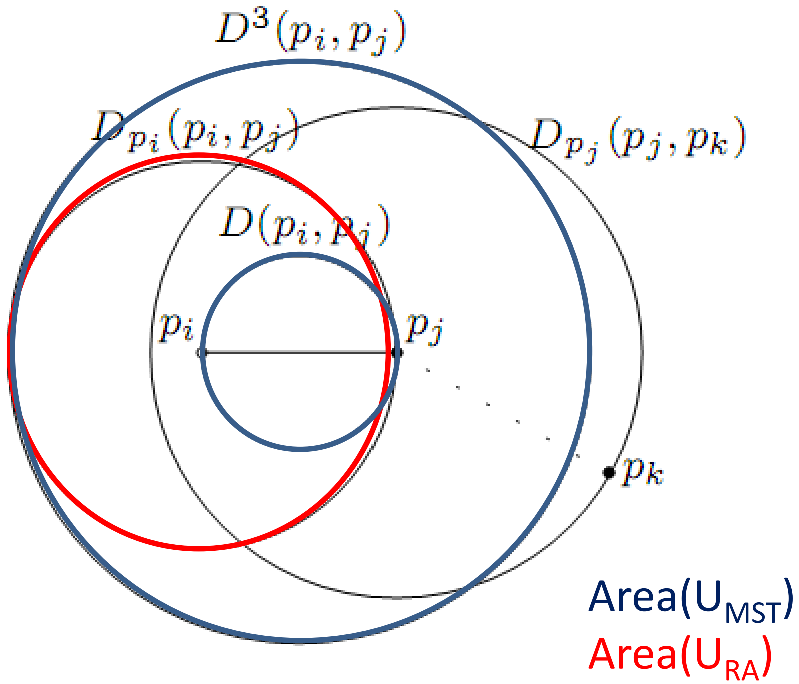


Fig. 5. $(p_i, p_j) \in$ MST; $D(p_i, p_j) \in D($MST$)$; $D_{p_i}(p_i, p_j), D_{p_j}(p_j, p_k) \in$ RA; $D^3(p_i, p_j) \in D^3($MST$)$.

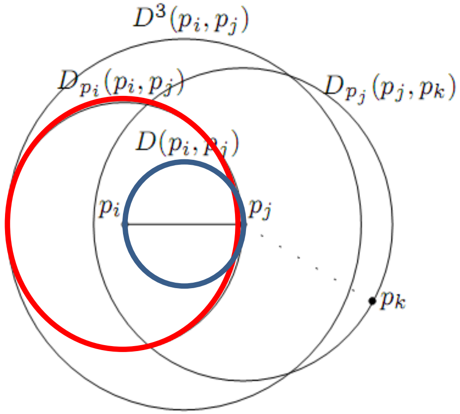
Claim 6: area(U_{RA}) ≤ 9·area(U_{MST})

- Proof
- The area($D_{pi}(p_i,p_j)$) ≤ area($D^3(p_i,p_j)$) = 9·areaD(p_i,p_j)
- area(U_{RA}) ≤ 9area(U_{MST})



Theorem 2. RA is a constant-factor approximation for MARA, i.e., area(U_{RA}) ≤ c'·area(U_{OPT}^R), for some constant c'

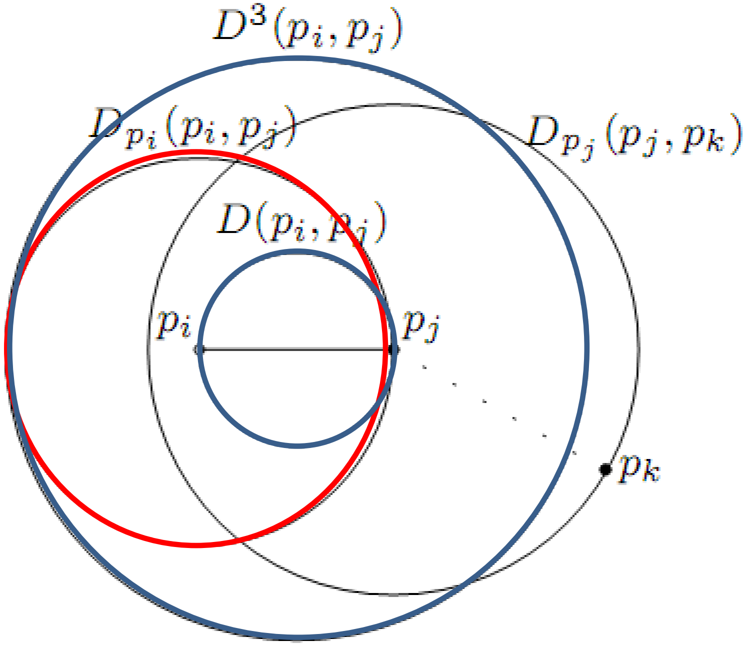
- Proof: area(U_{RA}) ≤ c'·area(U_{OPT}^R)
- We construct a spanning tree T of ***P*** as following,
  - For each point q ∈ P, q ≠p, compute a directed path from q to p, and add the
  - Make all edges in T undirected.
- Hence, $U_T \subseteq U_{OPT}^R$



area(U_{RA}) $\leq^1$ 9area(U_{MST}) $\leq^2$ 9c·area(U_{OPT}) $\leq^3$ 9c·area(U_T) $\leq^4$ 9c·area(U_{OPT}^R)

$$\text{area}(U_{RA}) \leq^1 9\text{area}(U_{MST}) \leq^2 9c\cdot\text{area}(U_{OPT}) \leq^3 9c\cdot\text{area}(U_T) \leq^4 9c\cdot\text{area}(U_{OPT^R})$$

- Inequality 1
  - According to claim 6 ($\text{area}(U_{RA}) \leq 9\cdot\text{area}(U_{MST})$)
- Inequality 2
  - Follows from Theorem 1 ($\text{area}(U_{MST})\leq c\cdot\text{area}(U_{OPT})$)
- Inequality 3
  - From the definition of OPT
- Inequality 4
  - Show above

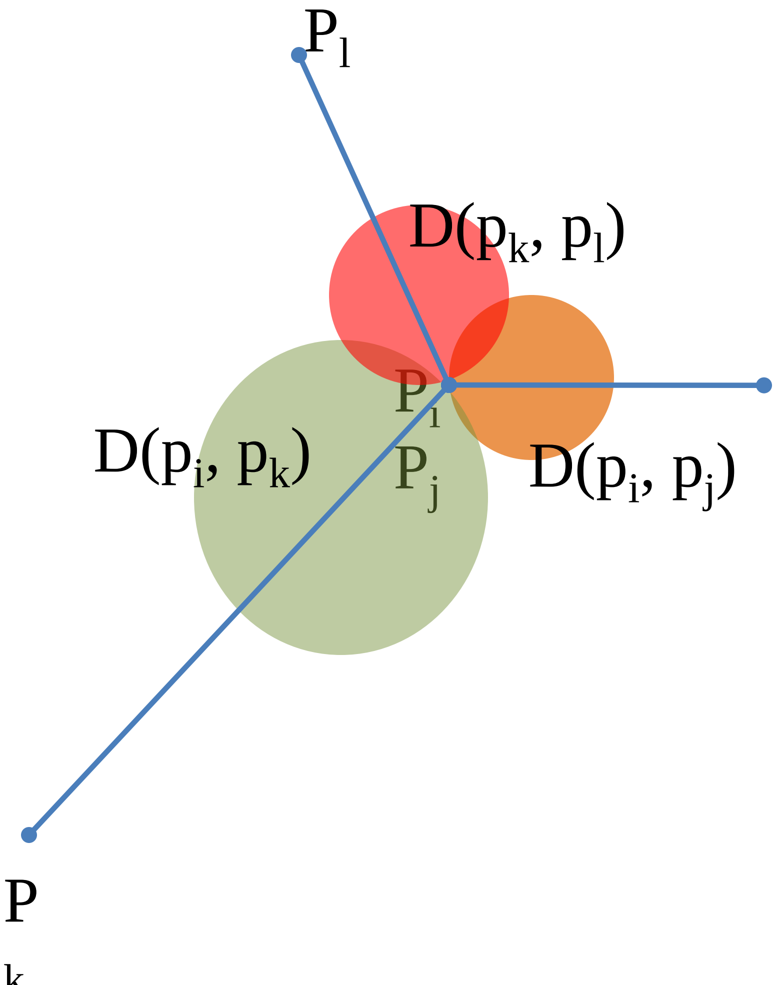


MACDG

A Constant-Factor Approximation for MACDG

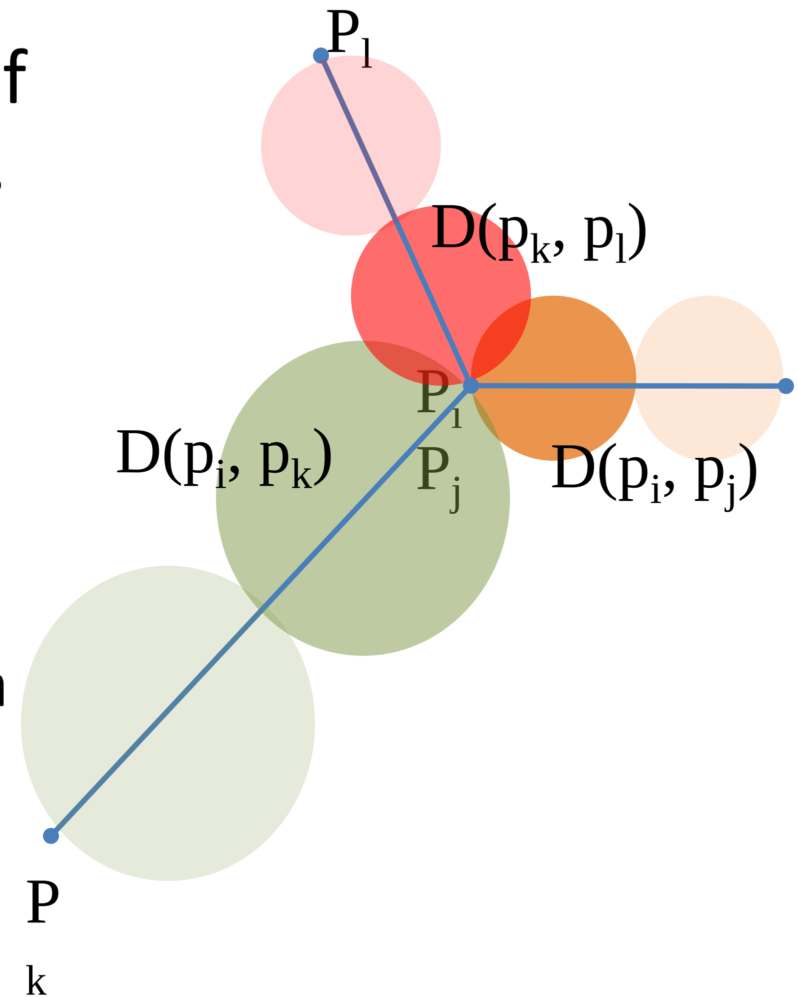
MACDG

- Minimum-Area Connected Disk Graph (MACDG) problem



MACDG: Goal and Define

- Goal:
  Decrease the coverage of overlapping from MARA.
- Define:
  - **DG = {D_{p1},..,D_{png}},** where D_{pi} is the disk of radius ri/2 centered at p_i
  - **OPT^D** denote an optimal assignment of radii, i.e., a solution to MACDG.



MACDG: A Constant-Factor Approximation

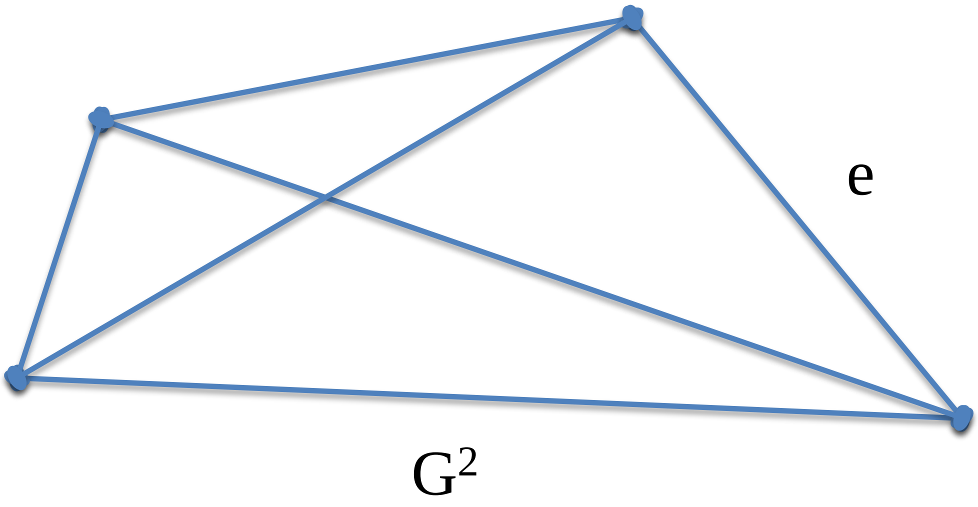
- Requirements:
  - (i) DG is connected
  - (ii) the area of the union of the disks in DG is bounded by some constant times the area of the union of the disks in an optimal assignment of radii
- The 1st requirement above clearly holds, since each edge in MST is also an edge in DG.
- And the 2nd requirement…(Theorem 3)

MACDG: Theorem 3

- DG is a constant-factor approximation for MACDG,
  $area(U_{DG}) \le c'' \cdot area(U_{OPT^D})$, for some constant c''
- Proof:
  - *(Claim 6)*
    It is very similar to the proof of MARA.
    Since $U_{DG} \subseteq U_{RA}$,
    $area(U_{DG}) \le 9 \cdot area(U_{MST})$
  - *(Theorem 1)*
    $area(U_{MST}) \le c \cdot area(U_{OPT^D})$
    $area(U_{DG}) \le 9 \cdot area(U_{MST}) \le 9c * area(U_{OPT^D}) = c'' \cdot area(UOPTD)$

Constant-Factor Approximation for MAT

Consider the complete graph induced by P, we assign the weights such that $w(e) = |e|^2$, i.e. the weight is the square of the length of the edge. Let G^2 denote this graph.



Relaxed Triangle Inequality

For points $u, v, w \in P$, triangle inequality $(|uv| \leq |\mathrm{uw}| + |\mathrm{wv}|)$ does not hold.

However, relaxed triangle inequality holds:

$$|uv|^2 \leq 2 \cdot (|uw|^2 + |wv|^2)$$

Constant-factor Approximation Algorithms for the TSP

- For distance functions:

  $$d(u,v) \leq \tau \cdot (|uw|^2 + |wv|^2)$$

- Andreae and Bandelt:

  $(3\tau^2/2+\tau/2)$-approximation

- Andrea: $(\tau^2+\tau)$-approximation
- Bender and Chekuri: 4τ-approximation
- This implies there is a 6-approximation for our case

Constant-Factor Approximation for MAT

- Andreae and Bandelt computed a tour T in G^2 such that $\mathrm{w}(T) \leq \mathrm{c} \cdot \mathrm{w}(\mathrm{MST}_{G^2})$
- T is a constant-factor approximation for the Minimum Area Tour (MAT) problem

Notations

- $D(e)$ denotes the disk whose diameter is e
- $D(T) = \{ D(e) \mid e$ is an edge in $T \}$
- $\bigcup_T = \bigcup_{e \in T} D(e)$
- $\sigma_T = \Sigma_{e \in T}\, \text{area}(D(e))$
- MST is the minimal spanning tree of P
- OPT^T is an optimal tour, i.e., a solution to MAT
- OPT^S is a solution to the MAST problem
- Clearly, $\text{area}(\bigcup_{\text{OPT}^T}) \geq \text{area}(\bigcup_{\text{OPT}^S})$

Proof

$$\text{area}(\bigcup_T) \le \sigma_T \le w(T) \le c \cdot w(\text{MST}_{G^2})$$

but $w(\text{MST}_{G^2}) = \sum_{e \in \text{MST}} |e|^2$

$$\text{area}(\bigcup_T) = O(\sum_{e \in \text{MST}} |e|^2) = O(\sigma_{\text{MST}}) = O(\text{area}(\bigcup_{\text{MST}}))$$

(by Lemma 1)

By the main result of section 2 (MAST)

$$O(\text{area}(\bigcup_{\text{MST}})) = O(\text{area}(\bigcup_{\text{OPT}^S})) = O(\text{area}(\bigcup_{\text{OPT}^T}))$$

Theorem 4

- T is a constant-factor approximation for MAT, i.e., $\text{area}(\cup_T) \leq \hat{c} \cdot \text{area}(\cup_{\text{OPT}^T})$