

# A note on Eccentricities, diameters, and radii\*

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Let  $G = (V, E, w)$  be a graph and  $U \subset V$ . By  $D_G(v, U)$ , we denote the maximum distance from vertex  $v$  to any vertex in  $U$ . For a vertex  $v$ , the *eccentricity* of  $v$  is the maximum of the distance to any vertex in the graph, i.e.,  $\max_{u \in V} \{d_G(v, u)\}$  or  $D_G(v, V)$ . The *diameter* of a graph is the maximum of the eccentricity of any vertex in the graph. (The term “diameter” is overloaded. It is defined as the maximum eccentricity and also as the path of length equal to the maximum eccentricity.) In other words, the diameter is the longest distance between any two vertices in the graph. Recall that the distance between two vertices is the length of their shortest path in the graph. It should not be confused with the longest path in the graph.

The *radius* of a graph is the minimum eccentricity among all vertices in the graph, and a *center* of a graph is a vertex with eccentricity equal to the radius. For a general graph, there may be several centers and a center is not necessarily on a diameter. For example, in Figure 1(a), the shortest path between  $v_1$  and  $v_4$  is a diameter of length 6. Meanwhile  $v_2$  and  $v_3$  are endpoints of another diameter. The four vertices represented by white circles are centers of the graph, and the radius is 4. Note that the centers are not on any diameter. The diameter, radius and center of a graph can be found by computing the distances between all pairs of vertices. It takes  $O(|V||E| + |V|^2 \log |V|)$  time for a general graph.

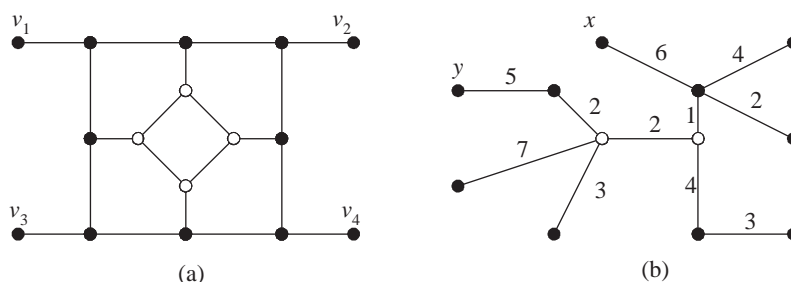


Figure 1: Diameters, centers and radii of (a) an unweighted graph; and (b) a weighted tree.

Any pair of vertices has a unique simple path in a tree. For this reason, the diameter, radius and centers of a tree are more related. For an unweighted tree  $T = (V, E)$ , it can be easily verified that

$$2 \times \text{radius} - 1 \leq \text{diameter} \leq 2 \times \text{radius}. \quad (1)$$

For positive weighted tree  $T = (V, E, w)$ , we can also have

$$2 \times \text{radius} - \max_e \{w(e)\} \leq \text{diameter} \leq 2 \times \text{radius}. \quad (2)$$

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\*An excerpt from the book “Spanning Trees and Optimization Problems,” by Bang Ye Wu and Kun-Mao Chao (2004), Chapman & Hall/CRC Press, USA.

Figure 1(b) illustrates an example of the diameter, radius, and center of a tree. The two vertices represented by white circles are the centers of the tree. There are four diameters of length 16 in the tree. Vertices  $x$  and  $y$  are the endpoints of a diameter. The radius is 9, and the centers are on the diameters.

More efficient algorithms for computing diameter, radius, and centers are available if the graph is a tree. Let  $T = (V, E, w)$  be a rooted tree. By  $T_r$ , we denote the subtree rooted at vertex  $r \in V$ , which is the subgraph induced on vertex  $r$  and all its descendants. Let  $child(r)$  denote the set of children of  $v$ . The eccentricity of the root of a tree can be computed by the following recurrence relation.

$$D_{T_r}(r, V(T_r)) = \max_{s \in child(r)} \{D_{T_s}(s, V(T_s)) + w(r, s)\}. \quad (3)$$

By a recursive algorithm or an algorithm visiting the vertices in postorder, the eccentricity can be computed in linear time since each vertex is visited once. To make it clear, we give a recursive algorithm in the following.

**Algorithm:** ECCENT( $T_r$ )

**Input:** A tree  $T_r = (V, E, w)$  rooted at  $r$ .

**Output:** The eccentricity of  $r$  in  $T_r$ .

- 1: **if**  $r$  is a leaf **then**  
    **return** 0;
- 2: **for** each child  $s$  of  $r$  **do**  
    compute ECCENT( $T_s$ ) recursively;
- 3: **return**  $\max_{s \in child(r)} \{ECCENT(s) + w(r, s)\}$ .

To find the eccentricity of a vertex in an unrooted tree, we can root the tree at the vertex and employ the ECCENT algorithm. Therefore, we have the next lemma.

**Lemma 1:** The eccentricity of a vertex in a tree can be computed in linear time.

Let  $x, y$  and  $z$  be three vertices in a tree  $T$ . It can be easily verified that the three paths  $SP_T(x, y)$ ,  $SP_T(x, z)$  and  $SP_T(y, z)$  intersect at a vertex. Define  $c(x, y, z)$  to be the intersection vertex. We have

$$d_T(x, c(x, y, z)) = \frac{1}{2} (d_T(x, y) + d_T(x, z) - d_T(y, z)), \quad (4)$$

or equivalently

$$d_T(x, c(x, y, z)) = \frac{1}{2} (d_T(x, y) + d_T(x, z) + d_T(y, z)) - d_T(y, z). \quad (5)$$

We now derive some properties to help us find the diameter of a tree.

**Fact 1:** Suppose that  $SP_T(v_1, v_2)$  is a diameter of  $T$  and  $r$  is a vertex on the diameter. For any vertex  $x$ ,  $d_T(x, r) \leq \max\{d_T(r, v_1), d_T(r, v_2)\}$ .

**Proof:** Otherwise  $SP_T(x, v_1)$  or  $SP_T(x, v_2)$  is a path longer than the diameter. □

The property can be easily extended to the case where  $r$  is not on the diameter. Let  $u = c(r, v_1, v_2)$ . Without loss of generality, let  $d_T(u, v_1) \geq d_T(u, v_2)$ . For any vertex  $x$ , we have  $d_T(x, u) \leq d_T(v_1, u)$  by Fact 1. Then

$$d_T(x, r) \leq d_T(x, u) + d_T(u, r) \leq d_T(v_1, u) + d_T(u, r) = d_T(v_1, r),$$

which implies that  $v_1$  is the farthest vertex to  $r$ . We can conclude that, for any vertex, one of the endpoints of a diameter must be the farthest vertex. Furthermore the converse of the property is also true.

Let  $r$  be any vertex in a tree and  $v_3$  be the vertex farthest to  $r$ . We shall show that  $v_3$  must be an endpoint of a diameter. Suppose that  $SP_T(v_1, v_2)$  is a diameter. By the above property, one of the endpoints, say  $v_1$ , is the farthest vertex to  $r$ , i.e.,

$$d_T(r, v_1) = d_T(r, v_3) \geq d_T(r, v_2).$$

Let  $u_1 = c(r, v_1, v_2)$  and  $u_2 = c(r, v_1, v_3)$ . Then  $u_2$  must be on the path  $SP_T(v_1, u_1)$ , for otherwise

$$\begin{aligned} d_T(v_1, v_3) &= d_T(v_1, u_2) + d_T(u_2, v_3) \\ &= 2d_T(v_1, u_2) > 2d_T(v_1, u_1) \\ &> d_T(v_1, v_2), \end{aligned}$$

a contradiction. If  $u_1 = u_2$ , then  $d_T(v_3, v_1) = d_T(v_1, v_2)$  and  $SP_T(v_3, v_1)$  is also a diameter. Otherwise,  $d_T(v_3, v_2) = d_T(v_1, v_2)$  and  $SP_T(v_3, v_2)$  is also a diameter. We have the next lemma.

**Lemma 2:** Let  $r$  be any vertex in a tree  $T$ . If  $v$  is the farthest vertex to  $r$ , the eccentricity of  $v$  is the diameter of  $T$ .

The following algorithm uses the property to find the diameter of a tree.

**Algorithm:** TREEDIAMETER

**Input:** A tree  $T = (V, E, w)$ .

**Output:** The diameter of  $T$ .

- 1: Root  $T$  at an arbitrary vertex  $r$ .
- 2: Use ECCENT to find the farthest vertex  $v$  to  $r$ .
- 3: Root  $T$  at  $v$ .
- 4: Use ECCENT to find the eccentricity of  $v$ .
- 5: Output the eccentricity of  $v$  as the diameter of  $T$ .

It is obvious that the algorithm runs in linear time. The radius and the center can be obtained from a diameter. Suppose that  $P = SP_T(v_1, v_2)$  is a diameter. Starting at  $v_1$  and traveling along the path  $P$ , we compute the distance  $d_T(u, v_1)$  for each vertex  $u$  on the path. Let  $u_1$  be the last encountered vertex such that  $d_T(v_1, u_1) \leq \frac{1}{2}w(P)$  and  $u_2$  be the next vertex to  $u_1$  (Figure 2(a)). By the definition of  $u_1$ ,  $u_2$  is the first encountered vertex such that  $d_T(v_1, u_2) > \frac{1}{2}w(P)$ . We claim that  $u_1$  or  $u_2$  is a center of the tree. Let  $P_1 = SP_T(v_1, u_1)$  and  $P_2 = SP_T(u_2, v_2)$ . First, by Fact 1, the eccentricities of  $u_1$  and  $u_2$  must be  $d_T(u_1, v_2)$  and  $d_T(u_2, v_1)$  respectively. Otherwise  $P$  cannot be a diameter. For any vertex  $x$  connected to  $P$  at a vertex in  $P_1$ , we have  $d_T(x, v_2) > d_T(u_1, v_2)$ . Similarly, for any vertex  $x$  connected to  $P$  at a vertex in  $P_2$ , we have  $d_T(x, v_1) > d_T(u_2, v_1)$ . Consequently the eccentricity of any vertex is at least  $\min\{d_T(u_1, v_2), d_T(u_2, v_1)\}$ , and  $u_1$  or  $u_2$  must be a center. Therefore the center as well as the radius of a tree can be computed in linear time since the diameter can be found in linear time.

**Theorem 3:** The diameter, radius, and center of a tree can be computed in linear time.

Let us examine some more properties of the diameters of a tree. We assume that all edge lengths are positive.

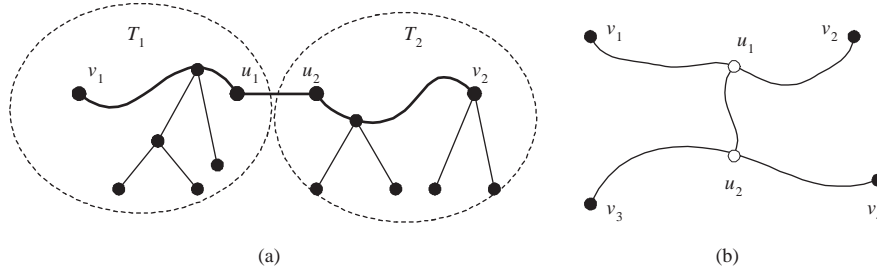


Figure 2: (a) Finding a center on a diameter. (b) Two diameters cannot be disjoint.

**Fact 2:** Two diameters of a tree cannot be disjoint.

**Proof:** Suppose that  $SP_T(v_1, v_2)$  and  $SP_T(v_3, v_4)$  are two disjoint diameters of a tree  $T$ . Let  $u_1 = c(v_3, v_1, v_2)$  and  $u_2 = c(v_1, v_3, v_4)$  (Figure 2(b)). We have

$$\begin{aligned} d_T(v_1, v_3) + d_T(v_2, v_4) &= d_T(v_1, v_2) + d_T(v_3, v_4) + 2d_T(u_1, u_2) \\ &> 2d_T(v_1, v_2) \end{aligned}$$

since  $d_T(v_1, v_2) = d_T(v_3, v_4)$  is the diameter and  $d_T(u_1, u_2) > 0$ . It implies that the path from  $v_1$  to  $v_3$  or the path from  $v_2$  to  $v_4$  is longer than the diameter, a contradiction.  $\square$

Let  $\mathcal{P}$  be a set of more than two paths of a tree and the paths intersect each other. One can easily verify that all the paths in  $\mathcal{P}$  share a common vertex. Otherwise there exists a cycle, which contradicts the definition of a tree. Therefore we can have the next property.

**Fact 3:** All diameters of a tree share at least one common vertex.