Yet Another Proof.

Kun-Mao Chao

Thm. Let n be a positive number. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either increasing or decreasing.

pf. $(a_1, a_2, \cdots, a_{n^2+1})$

Let inc_i be the length of the longest increasing subsequence starting at a_i for $1 \le i \le n^2+1$.

Assume that $inc_i \le n$ for $1 \le i \le n^2+1$.

Otherwise we have an increasing subsequence of length $n+1$.

Since there are n^2+1 numbers, by the pigeonhole principle, there exists $1 \le k \le n$, such that we can find at least $n+1$ number, say $a_{i_1}, a_{i_2}, \cdots, a_{i_{n+1}}$, where $i_1 < i_2 < \cdots < i_{n+1}$, and

$$inc_{i_1} = inc_{i_2} = \cdots = inc_{i_{n+1}} = k.$$

If $a_{i_j} < a_{i_{j+1}}$, then $inc_{i_j} > inc_{i_{j+1}}$.

Thus, $a_{i_1} > a_{i_2} > \cdots > a_{i_{n+1}}$.

a decreasing subsequence of length $n+1$.

Q.E.D.

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