Ex. $L = \{w \in \{a,b\}^*:$ Kim Hao Chan

CFG vs. Pushdown Automata

w has the same number of a's and b's $\}$

$abbbabaa \in L$

CFG:

$S \to aSbS$

$S \to bSaS$

$S \to e$

$L \cap a^*b^* = \{a^nb^n : n \geq 0\}$

regular not regular

not regular

$a \underbrace{\ldots}_{\#a=\#b} b \underbrace{\ldots}_{\#a=\#b}$

$S \qquad S$

$b \underbrace{\ldots}_{\#a=\#b} a \underbrace{\ldots}_{\#a=\#b}$

$S \Rightarrow aSbS \Rightarrow abS$
$\Rightarrow abbSaS \Rightarrow abbSa$
$\Rightarrow abbbSaSa$
$\Rightarrow abbbSaa \Rightarrow abbbaSbSaa$
$\Rightarrow abbbabSaa \Rightarrow abbbabaa$

$a{:}+1$
$b{:}-1$

$a \mid a\,a\,b\,a\,b\,a\,b\,b\,a\,b\,b\,a\,b$

$+1 \mid +2\ +3\ +2\ +3\ +2\ +3\ +2\ +1\ +2\ +1\ +0\ +1\ +0$

$\underset{a}{\#\text{of } +1} < \underset{b}{\#\text{of } -1}$ $\exists +0 \Rightarrow \underset{+1}{a} \overbrace{\ldots}^{\#a=\#b} \underset{+0}{b} \overbrace{\ldots}^{\#a=\#b}$

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$L = \{ w \in \{a,b\}^* :$
w has the same number
of a's and b's $\}$

$abbbabaa \in L$

Yet Another
CFG:

$S \to SAB$

$S \to e$

$A \to aSb$

$A \to e$

$B \to bSa$

$B \to e$

B
bSa B A
$abbbaabaab$
A

$S \Rightarrow SAB \Rightarrow \cdots \Rightarrow SABABAB\cdots AB$

$S \overset{*}{\Rightarrow} SABABAB$

$S \Rightarrow e$, $A \overset{*}{\Rightarrow} ab$, $B \Rightarrow bSa \overset{*}{\Rightarrow} ba$, $A \overset{*}{\Rightarrow} e$, $B \overset{*}{\Rightarrow} ba$, $A \overset{*}{\Rightarrow} ab$, $B \overset{*}{\Rightarrow} e$

$\Rightarrow abbbaabaab$

Ex. $L = \{ w \in \{a,b\}^* :$ Kun-Mao Chao

w has the same number of a's and b's $\}$

$abbbabaa \in L$

CFG:

$S \to aA$ ← one more a, asking for one b

$S \to bB$ ← one more b, asking for one a

$S \to e$

$A \to bS$ ← got one b, back to S

$A \to aAA$ ← got one a, two more A's (a's & then A's)

$B \to aS$ ← got one a, back to S

$B \to bBB$ ← got one b, two more b's & then B's

$$S \Rightarrow aA \Rightarrow abS \Rightarrow abbB \Rightarrow abbbBB \Rightarrow abbbaSB$$
$$\Rightarrow abbbabBB \Rightarrow abbbabaSB \Rightarrow abbbabaB$$
$$\Rightarrow abbbabaaS \Rightarrow abbbabaa$$

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Pushdown automaton:

$M = (K, \Sigma, \Gamma, \Delta, s, F)$

$K = \{s, q, f\}$

$\Sigma = \{a, b\}$

$\Gamma = \{a, b, c\}$

$F = \{f\}$

Δ: $((s, e, e), (q, c))$ push c

$((q, a, c), (q, ac))$ push a

$((q, a, a), (q, aa))$ push a

$((q, a, b), (q, e))$ pop b

$((q, b, c), (q, bc))$ push b

$((q, b, b), (q, bb))$ push b

$((q, b, a), (q, e))$ pop a

$((q, e, c), (f, e))$ pop c

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state	remaining input	stack
s	abbbabaa	e
q	abbbabaa	c
q	bbbabaa	ac
q	bbabaa	c
q	babaa	bc
q	abaa	bbc
q	baa	bc
q	aa	bbc
q	a	bc
q	e	c
f	e	e

Ex. Finite automata $\quad (p, u, q) \in \Delta$

$\Downarrow \qquad\qquad \Downarrow$

Pushdown automata $\quad ((p, u, e)(q, e)) \in \Delta'$

no pushdown operation

CFG $\Rightarrow$ Pushdown automata

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Ex. CFG:

$G = (V, \Sigma, R, S)$

$V = \{S, a, b, c\}$

$\Sigma = \{a, b, c\}$ abbcbba

R:

$S \to aSa$ $S \Rightarrow aSa \Rightarrow abSba$

$S \to bSb$ $\Rightarrow abbSbba \Rightarrow abbcbba$

$S \to c$ $L(G) = \{wcw^R : w \in \{a,b\}^*\}$

Pushdown automaton:

$M = \{K, \Sigma, \Gamma, \Delta, S, F\}$

$K = \{s, q\}$, $\Sigma = \{a, b, c\}$, $\Gamma = V = \{S, a, b, c\}$, $F = \{q\}$.

$\Delta = \{((s,e,e),(q,S)),$
$((q,e,S),(q,aSa)),$
$((q,e,S),(q,bS,b)),$
$((q,e,S),(q,c)),$
$((q,a,a)),(q,e)),$
$((q,b,b)),(q,e)),$
$((q,c,c)),(q,e))\}$

state	input	stack
s	abbcbba	e
q	abbcbba	S
q	abbcbba	aSa
q	bbcbba	Sa
q	bbcbba	bSba
q	bcbba	Sba
q	bcbba	bSbba
q	cbba	Sbba
q	cbba	cbba
q	bba	bba
q	ba	ba
q	a	a
⋮	e	e

Thm. The context-free languages are closed under union, concatenation, and Kleene star.

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pf. $G_1 = (V_1, \Sigma_1, R_1, S_1)$; $G_2 = (V_2, \Sigma_2, R_2, S_2)$; $(V_1 - \Sigma_1) \cap (V_2 - \Sigma_2) = \phi$

nonterminals

otherwise, rename those nonterminals.

union:

$$(V_1 \cup V_2 \cup \{S\}, \Sigma_1 \cup \Sigma_2, \underbrace{R_1 \cup R_2 \cup \{S \to S_1, S \to S_2\}}, S)$$

concatenation:

$$R_1 \cup R_2 \cup \{S \to S_1 S_2\}$$

Kleene star:

$$L(G) = L(G_1)^*$$

$$G = (V_1 \cup \{S\}, \Sigma_1, R_1 \cup \{S \to SS_1, S \to e\}, S)$$

#

Note that the context-free languages are not closed under intersection or complementation. We'll prove this after introducing the pumping theorem. Coming right up.

Pumping Theorem

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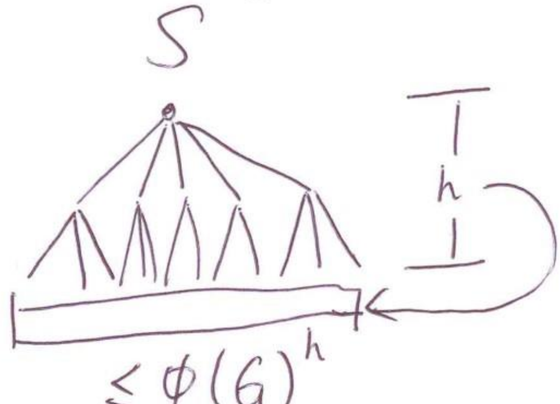
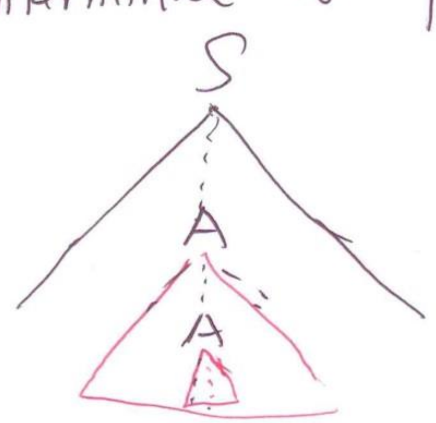
$$G = (V, \Sigma, R, S)$$

Let $\phi(G)$ be the largest number of symbols on the right-hand side of any rules in R.



In the pumping lemma for r.e., we use the path between the same state to "pump" more symbols.

In the pumping theorem for CFL, we use the path between the same nonterminal to "pump" more symbols.



How?

Thm. Let (V, Σ, R, S) be a context-free grammar. Then any string $w \in L(G)$ of length greater than $\phi(G)^{|V-\Sigma|}$ can be rewritten as $w = uvxyz$ in such a way that either v or y is nonempty and $uv^n x y^n z \in L(G)$ for every $n \geq 0$.

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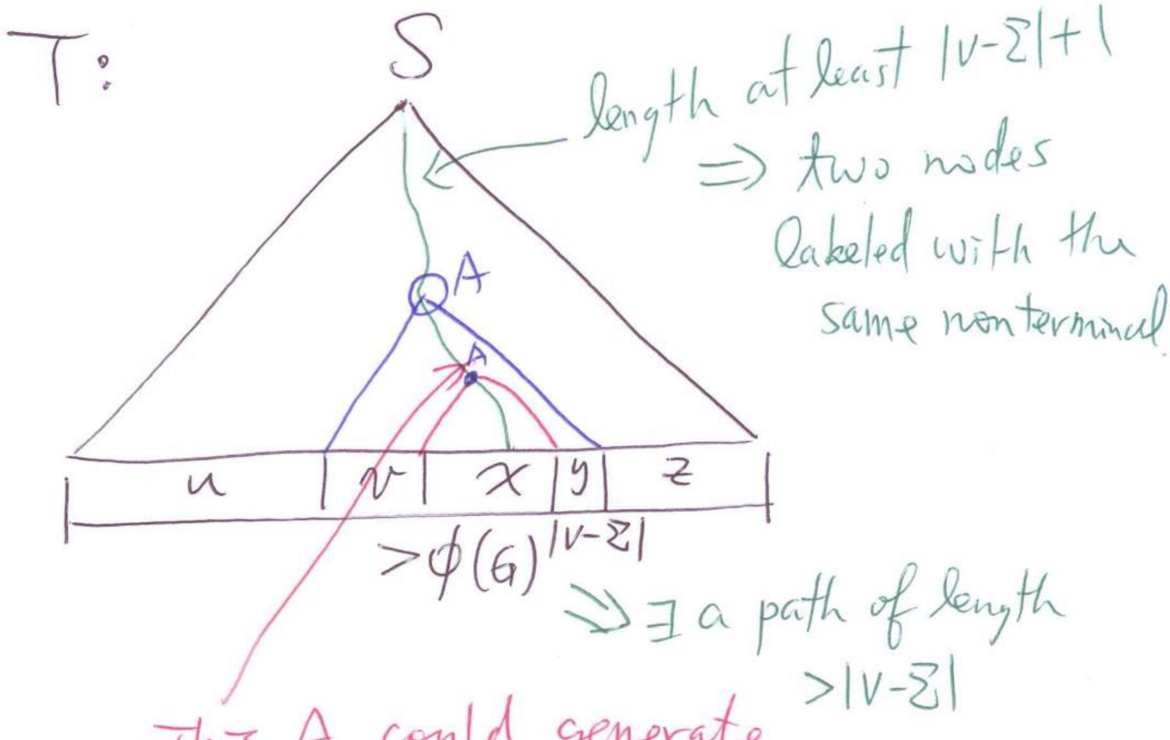
Pf. Let T be the parse tree with root labeled S and with yield w that has the smallest number of leaves among all parse trees with root S and yield w.



This A could generate $x, vxy, v^2xy^2, v^3xy^3, \ldots$

If $vy = e$, T is not smallest. A contradiction

※

Ex. $L = \{ a^n b^n c^n : n \geq 0 \}$.

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Suppose that $L = L(G)$ for some context-free grammar $G = (V, \Sigma, R, S)$

Let $n > \frac{\phi(G)^{|V - \Sigma|}}{3}$.

By the pumping theorem, $w = a^n b^n c^n$ can be rewritten as $w = uvxyz$ such that $vy \neq e$ and

$$u v^n x y^n z \in L(G) \text{ for } n = 0, 1, 2, \cdots$$

If vy contains all elements in $\{a, b, c\}$ then at least one of v, y must contain at least two elements in $\{a, b, c\}$.

$$\Rightarrow u \underbrace{v^2}_{abab} x y^2 z \notin L(G)$$

If vy contains some but not all elements in $\{a, b, c\}$,

then $uv^2xy^2z \notin L(G)$

unequal numbers of a's, b's, and c's.

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$L_1 = \{a^n b^n c^m : m, n \geq 0\}$

CFG for L_1

$$\left.\begin{array}{l} S \to XY \\ X \to aXb \\ X \to e \end{array}\right] a^n b^n$$

$$\left.\begin{array}{l} Y \to Yc \\ Y \to e \end{array}\right] c^m$$

$L_2 = \{a^m b^n c^n : m, n \geq 0\}$

Both L_1 and L_2 are CFL's.

$L_1 \cap L_2 = \{a^n b^n c^n : n \geq 0\}$ is not a CFL.

$\Rightarrow$ The context-free languages are not closed under intersection.

If the context-free languages are closed under complementation

$$\left.\overline{\underset{\text{CFL}}{\overline{L_1}} \cup \underset{\text{CFL}}{\overline{L_2}}}\right) \text{CFL} \Rightarrow L_1 \cap L_1 \text{ is CFL.}$$ A contradiction.

$\Rightarrow$ CFL's are not closed under