Sets, Relations, and Functions

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- A set is a collection of objects.

  e.g. $\{1, 4, 7\}$, $\{b, d, f\}$.

- The power set of a set A, denoted by 2^A, is the collection of all subsets of A.

  e.g. $2^{\{b,d,f\}} = \{\phi, \{b\}, \{d\}, \{f\}, \{b,d\}, \{b,f\}, \{d,f\}, \{b,d,f\}\}$.

- The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$.

e.g. $\{1,4,7\} \times \{b,d,f\} = \{(1,b),(1,d),(1,f),(4,b),(4,d),(4,f),(7,b),(7,d),(7,f)\}$

△ What is a partition of a set?

We say that $\Pi = \{\pi_1, \pi_2, \cdots, \pi_k\}$ is a partition of a set A if

(1) $\pi_i \neq \phi \quad 1 \le i \le k$;

(2) $\pi_i \cap \pi_j = \phi \quad 1 \le \underline{i \neq j} \le k$;

(3) $\bigcup_{i=1}^{k} \pi_i = A$.

3 bins

ex. $A = \{a, b, c, d, e\}$.

✓ $\{\{a, c, d\}, \{b, e\}\}$, ✓ $\{\{a, b\}, \{c, d\}, \{e\}\}$

2 bins

× $\{\{a, b\}, \{c, d\}\}$, × $\{\{a, b, c\}, \{c, d, e\}\}$

missing $\{e\}$ duplicated

△ Let A be the set of students in this class. What is the number of partitions of A?

Hint: Count with the number of bins fixed.

$S(n,k)$: The number of partitions of a set of n numbers with k bins.

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$

$$S(1,1) = 1$$
$$S(*,1) = 1$$

$$S(2,2) = 1$$
$$S(3,3) = 1 \qquad S(k,k) = 1$$

$$S(3,2) = S(2,1) + 2 \cdot S(2,2)$$
$$= 1 + 2 = 3$$

$$S(4,2) = S(3,1) + 2 \cdot S(3,2)$$
$$= 1 + 2 * 3 = 7$$

$$S(5,2) = S(4,1) + 2 \cdot S(4,2)$$
$$= 1 + 2 * 7 = 15$$

$$S(6,2) = S(5,1) + 2 * S(5,2)$$
$$= 1 + 2 * 15 = 31$$

...

B_n: the number of partitions of a set of n elements.

$S(n,k)$: the number of partitions of a set of n elements with k bins.

$$B_n = \sum_{k=1}^{n} S(n,k)$$

Now we show how to compute B_{10}.

$$B_{10} = \overset{=1}{S(10,1)} + S(10,2) + S(10,3) + \cdots + \overset{=\binom{10}{2}=45}{S(10,9)} + \overset{=1}{S(10,10)}$$

Let $A = \{a_1, a_2, \cdots, a_{10}\}$.

$$S(10,k) = \overset{\{a_1\} \cup \overbrace{\cdots}^{k-1}}{S(9,k-1)} + \overset{\overbrace{\{a_1, \cdots\} \cup \cdots}^{k}}{k\,S(9,k)}$$

S

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	B_n
1	1										1
2	1	1									2
3	1	3	1								5
4	1	7	6	1							15
5	1	15	25	10	1						52
6	1	31	90	65	15	1					203
7	1	63	301	350	140	21	1				877
8	1	127	966	1701	1050	266	28	1			4140
9	1	255	3025	7770	6951	2646	462	36	1		21147
10	1	511	9330	34105	42525	22827	5880	750	45	1	115975 ← B_{10}

You may compute B_n by another formula: $B_n = \sum_{j=1}^{n} \underset{|\{a_1,\cdots\}|=j}{\binom{n-1}{j-1}} B_{n-j} = \sum_{j=0}^{n-1} \binom{n-1}{j} B_j$ $\quad |\{a_1,\cdots\}| = n-j$

oeis.org/A000110

Let $A_1, \dots, A_n$ be any sets.

n-fold Cartesian product

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) : a_i \in A_i, \text{ for each } i = 1, \dots, n\}$$

An n-ary relation on sets $A_1, \dots, A_n$ is a subset of $A_1 \times \cdots \times A_n$.

1-ary : unary
2-ary : binary
3-ary : ternary

domain

image

A function f from a set A to a set B, denoted as $f : A \mapsto B$, can be viewed as a binary relation where for each $a \in A$, there is exactly one ordered pair in R with first component a.

one-to-one : $a, a' \in A \Rightarrow f(a) \neq f(a')$;
onto : $\forall\, b \in B, \exists\, a \in A$ s.t. $f(a) = b$;
bijection : one-to-one & onto.

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Let us assume that

(a). Each one is a friend of him/her-self;

(b) If x is a friend of y, y is a friend of x;

(c). A friend of a friend is a friend.

Consider the binary relation of a set A:

$R = \{(x,y): x,y \in A \text{ and } x \text{ is a friend of } y\}$.

Let $A = \{a,b,c,d,e,f,g,h\}$.

(a) $(a,a) \in R, \ldots$ reflexive; (b) $(a,b) \in R \Rightarrow (b,a) \in R$; symmetric

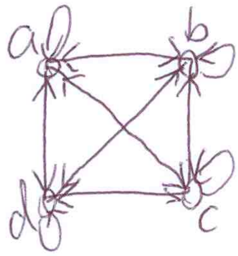
(c) If $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R$. transitive

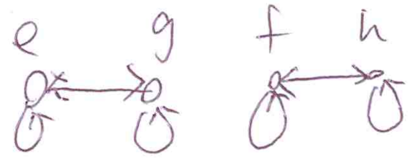
Equivalence relation

reflexive, symmetric, transitive

$R = \{\overset{[a]}{\{a,b,c,d\}}, \overset{[e]}{\{e,g\}}, \overset{[f]}{\{f,h\}}\}$

Equivalence classes $\Updownarrow$ a partition.





Assume that $[a] \neq [e]$ & $[a] \cap [e] \neq \phi$

$x \in [a] \Rightarrow (x,a) \in R$

$x \in [e] \Rightarrow (x,e) \in R$

$\Rightarrow (a,e) \in R$

For any $y \in [a]$, $(y,a) \in R$

$\Rightarrow (y,e) \in R \Rightarrow y \in [e]$

$\Rightarrow [a] \subseteq [e]$

Similarly, $[e] \subseteq [a]$.

We have $[a] = [e]$.

A contradiction.