Some Proof Techniques.

Kun-Mao Chao

△ Proof by contradiction

Prove that $\sqrt{2}$ is irrational.

Pf. Let $\sqrt{2} = \frac{m}{n}$, where $m \geq 1$ and $n \geq 1$.

We assume that m and n are not both even. (For otherwise, ...)

$$\sqrt{2} = \frac{m}{n} \Rightarrow 2n^2 = m^2$$

$$\Rightarrow m^2 \text{ is even}$$

$$\Rightarrow m \text{ is even}$$

Let $m = 2k$

$$2n^2 = (2k)^2 = 4k^2$$

$$n^2 = 2k^2$$

$$\Rightarrow n \text{ is even}$$

A contradiction.

We conclude that $\sqrt{2}$ is irrational.

Kun-Mao Chao

△ Nonconstructive proofs

Prove that $\exists$ irrational numbers x and y such that x^y is rational, i.e., $x^y \in Q$.

Proof. If $\sqrt{2}^{\sqrt{2}} \in Q$, we are done. $(x = \sqrt{2}, y = \sqrt{2})$.

Otherwise, $\sqrt{2}^{\sqrt{2}} \notin Q$. $\left(\underbrace{\sqrt{2}^{\sqrt{2}}}_{x}\right)^{\sqrt{2} \leftarrow y} = \sqrt{2}^{2} = 2$.

$(x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2})$ #

You may use $\sqrt{3}^{\sqrt{3}}$, $\cdots$ as well.

△ Proof by Induction

Kun-Mao Chao

Prove that for any finite set A, $|2^A| = 2^{|A|}$.

e.g.

$$|2^{\{b,d,f\}}| = |\{\phi, \{b\}, \{d\}, \{f\}, \{b,d\}, \{b,f\}, \{d,f\}, \{b,d,f\}\}| = 8 = 2^3 = 2^{|\{b,d,f\}|}$$

proof.

Basis Step. $|A| = 0 \Rightarrow A = \phi$

$$|2^A| = |\{\phi\}| = 1 = 2^0 = 2^{|A|}.$$

Induction Hypothesis.

Suppose that $|2^A| = 2^{|A|}$ for $|A| \leq n$.

Induction Step.

Let $|A| = n+1$, and $a \in A$.

$B = A - \{a\} \Rightarrow |B| = n$

$|2^B| = 2^{|B|} = 2^n$

$|2^B|$

$2^A = 2^B \cup \{C \cup \{a\} : C \in 2^B\}$

$|2^A| = 2^n + 2^n = 2^{n+1} = 2^{|A|}$ Q.E.D.

Kai-Min Chung

△ The pigeon hole principle.

Thm. Let n be a positive number. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either increasing or decreasing.

e.g. $(18, \underline{5}, 20, \underline{8}, \underline{19})$

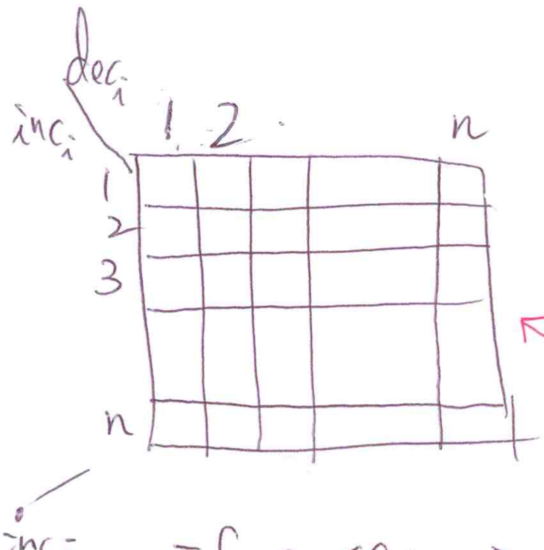
$(a_1, a_2, \cdots, a_{n^2+1})$

inc_i: the length of the longest increasing subsequence starting at a_i.

dec_i: " decreasing "

Assume that $inc_i \le n$ & $dec_i \le n$.

(pigeons)



← n^2+1 numbers

(pigeon holes)

← n^2 buckets

$\exists\, i<j$ s.t. $inc_i = inc_j$, $dec_i = dec_j$

inc_j

$\cdots, a_i, \cdots, a_j \cdots$

dec_j

If $a_i < a_j \Rightarrow inc_i \ge 1 + inc_j$

If $a_i > a_j \Rightarrow dec_i \ge 1 + dec_j$.

Kun-Mao Chao

A is finite if $\exists$ a bijection function $f: A \mapsto \{1, 2, \cdots, n\}$ for some $n \in N$.

If A is not finite, it is infinite.

A is countably infinite if $\exists$ a bijection function $f: A \mapsto N$. [Note that $N = \{0, 1, 2, \cdots\}$ in this book.]

A is countable if it is finite or countably infinite.

Eg. The set of NTUCSIE teachers and students is countable. [finite].

The set of positive even numbers is countable. [$f(2)=0, f(4)=1, \cdots, f(i)=\frac{i}{2}-1, \cdots$]

The set of positive rational numbers is countable. [Why? Give it a try before you turn to the next page.]

△ Proof by Enumeration

Kun-Mao Chao

The set of positive rational numbers is countable.

Let's count.

$$
\begin{array}{cccc}
{}^{0}\,\frac{1}{1}\,{}^{0} & & & \\
{}^{1}\,\frac{1}{2}\,{}^{1} & {}^{2}\,\frac{2}{1}\,{}^{2} & & \\
{}^{3}\,\frac{1}{3}\,{}^{3} & \frac{2}{2}\,{}^{4} & {}^{4}\,\frac{3}{1}\,{}^{5} & \\
{}^{5}\,\frac{1}{4}\,{}^{6} & {}^{6}\,\frac{2}{3}\,{}^{7} & {}^{7}\,\frac{3}{2}\,{}^{8} & {}^{8}\,\frac{4}{1}\,{}^{9} \\
\vdots & & &
\end{array}
$$

Since there are duplicated rational numbers, you might want to skip counting $\frac{j}{i}$ if $\gcd(i,j) \neq 1$.

$$\frac{j}{i} \leftarrow ?\ \sum_{x=0}^{i+j-2} x + (j-1)$$

$$= \frac{(i+j-2)(i+j-1)}{2} + (j-1)$$

Let A, B, C be countable sets. $A = \{a_0, a_1, a_2, \ldots\}$,

$B = \{b_0, b_1, b_2, \ldots\}$, $C = \{c_0, c_1, c_2, \ldots\}$.

$A \cup B \cup C$ is countable.

$$
\begin{array}{lllll}
A & \cdot\, a_0\ 0 & \cdot\, a_1\ 3 & \cdot\, a_2\ 6 & \ldots \\
B & \cdot\, b_0\ 1 & \cdot\, b_1\ 4 & \cdot\, b_2\ 7 & \ldots \\
C & \cdot\, c_0\ 2 & \cdot\, c_1\ 5 & \cdot\, c_2\ 8 & \ldots
\end{array}
$$

Kun-Mao Chao

$\mathbb{N} \times \mathbb{N}$ (or $\mathbb{N}^2$) is countable.

$(0,0)^0$

$(0,1)^1 \ (1,0)^2$

$(0,2)^3 \ (1,1)^4 \ (2,0)^5$

$(0,3)^6 \ (1,2)^7 \ (2,1)^8 \ (3,0)^9$

$(0,4)^{10} \ (1,3)^{11} \ (2,2)^{12} \ (3,1)^{13} \ (4,0)^{14}$

$\vdots$

$$(i,j) \leftarrow ? \ \sum_{x=0}^{i+j} x + i = \frac{(i+j)(i+j+1)}{2} + i$$

$$= \frac{1}{2}\left[(i+j)^2 + 3i + j\right]$$

△ The Diagonalization Principle

The set of real numbers in $(0,1)$ is uncountable. #

Assume that it is countable.

$r_0 = 0.d_{00}\ d_{01}\ d_{02} \cdots$

$r_1 = 0.d_{10}\ d_{11}\ d_{12} \cdots$

$\vdots$

$r_n = 0.d_{n0}\ d_{n1} \cdots d_{nn} \cdots$

$\vdots$

$s = 0.s_0 s_1 s_2 \cdots \leftarrow s \neq r_i \ \forall i$

$$s_i = \begin{cases} 6 & \text{if } d_{ii} = 7 \\ 7 & \text{otherwise} \end{cases}$$

A contradiction.

Kam-Mau Chan

Power set: The collection of all subsets of a set A.

2^A

$$2^{\{a,b\}} = \{\phi, \{a\}, \{b\}, \{a,b\}\}$$

2^N is uncountable.

$$2^N = \{\phi, \{0\}, \{1\}, \{2\}, \dots, \{0, \cancel{0}\}, \{0,1\}, \{0,2\}, \dots, \{0, \cancel{0}\}, \{\cancel{0}, 0, 1\}, \dots\}$$

pf. Assume that 2^N is countable.

$$2^N = \{R_0, R_1, \dots\}$$

$$D = \{n \in N : n \notin R_n\}$$

$$D = R_k \Rightarrow \begin{cases} \text{if } k \in R_k \Rightarrow k \notin D \Rightarrow k \notin R_k. \\ \text{if } k \notin R_k \Rightarrow k \in D \Rightarrow k \in R_k. \end{cases}$$

A contradiction.