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A is finite if $\exists$ a bijection function $f: A \mapsto \{1, 2, \cdots, n\}$ for some $n \in N$.

If A is not finite, it is infinite.

A is countably infinite if $\exists$ a bijection function $f: A \mapsto N$. [Note that $N = \{\underline{0}, 1, 2, \cdots\}$ in this book.]

A is countable if it is finite or countably infinite.

Eg. The set of NTUCSIE teachers and students is countable. [finite].

The set of positive even numbers is countable. [$f(2)=0$, $f(4)=1$, $\cdots$, $f(i)=\frac{i}{2}-1$, $\cdots$]

The set of positive rational numbers is countable. [Why? Give it a try before you turn to the next page.]

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The set of positive rational numbers is countable.

Let's count.

$0\ \frac{1}{1}\ 0$

$1\ \frac{1}{2}\ 1, \quad 2\ \frac{2}{1}\ 2$

$3\ \frac{1}{3}\ 3 \quad \frac{2}{2}\ 4, \quad 4\ \frac{3}{1}\ 5$

$5\ \frac{1}{4}\ 6, \quad 6\ \frac{2}{3}\ 7, \quad 7\ \frac{3}{2}\ 8, \quad 8\ \frac{4}{1}\ 9$

$\vdots$

Since there are duplicated rational numbers, you might want to skip counting $\frac{j}{i}$ if $\gcd(i,j) \neq 1$.

$$\frac{j}{i} \leftarrow ? \sum_{x=0}^{i+j-2} x + (j-1)$$

$$= \frac{(i+j-2)(i+j-1)}{2} + (j-1)$$

Let A, B, C be countable sets. $A = \{a_0, a_1, a_2, \cdots\}$, $B = \{b_0, b_1, b_2, \cdots\}$, $C = \{c_0, c_1, c_2, \cdots\}$.

$A \cup B \cup C$ is countable.

A	a_0 0	a_1 3	a_2 6	$\cdots$
B	b_0 1	b_1 4	b_2 7	$\cdots$
C	c_0 2	c_1 5	c_2 8	$\cdots$

$N \times N$ (or N^2) is countable.

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$(0,0)^0$

$(0,1)^1 \ (1,0)^2$

$(0,2)^3 \ (1,1)^4 \ (2,0)^5$

$(0,3)^6 \ (1,2)^7 \ (2,1)^8 \ (3,0)^9$

$(0,4)^{10} \ (1,3)^{11} \ (2,2)^{12} \ (3,1)^{13} \ (4,0)^{14}$

$\vdots$

$$(i,j) \leftarrow ? \quad \sum_{x=0}^{i+j} x + i = \frac{(i+j)(i+j+1)}{2} + i$$

$$= \frac{1}{2}\left[(i+j)^2 + 3i + j\right]$$

#

The set of real numbers in $(0,1)$ is uncountable.

Assume that it is countable.

$r_0 = 0.d_{00}\ d_{01}\ d_{02} \cdots$

$r_1 = 0.d_{10}\ d_{11}\ d_{12} \cdots$

$\vdots$

$r_n = 0.d_{n0}\ d_{n1} \cdots d_{nn} \cdots$

$\vdots$

$S = 0.S_0 S_1 S_2 \cdots \leftarrow S \neq r_i \ \forall i$

A contradiction.

$$S_i = \begin{cases} 6 & \text{if } d_{ii} = 7 \\ 7 & \text{otherwise} \end{cases}$$

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Power set : The collection of all subsets of a set A.

2^A

$$2^{\{a,b\}} = \{\phi, \{a\}, \{b\}, \{a,b\}\}$$

2^N is uncountable.

$2^N = \{\phi, \{0\}, \{1\}, \{2\}, \dots, \{0,0\}, \{0,1\}, \{0,2\}, \dots, \{0,0,0\}, \{0,0,1\}, \dots\}$

pf.

Assume that 2^N is countable.

$$2^N = \{R_0, R_1, \dots\}$$

$$D = \{n \in N : n \notin R_n\}$$

$$D = R_k \Rightarrow \begin{cases} \text{if } k \in R_k \Rightarrow k \notin D \Rightarrow k \notin R_k \\ \text{if } k \notin R_k \Rightarrow k \in D \Rightarrow k \in R_k \end{cases}$$

A contradiction.

Closures.

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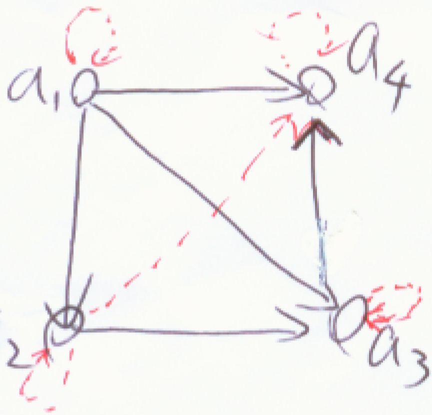
$R \subseteq A^2$: a directed graph defined on a set A ← $A \times A$

The reflexive transitive closure of R:

$$R^* = \{(a,b) : a, b \in A \text{ and } \exists \text{ a path from } a \text{ to } b \text{ in } R\}.$$



$R = \{(a_1,a_2), (a_1,a_3), (a_1,a_4), (a_2,a_3), (a_3,a_4)\}$

$R^* = R \cup \{(a_1,a_1), (a_2,a_2), (a_3,a_3), (a_4,a_4), (a_2,a_4)\}$

$A = \{a_1, a_2, \ldots, a_n\}$

How to compute R^*?

Initially $R^* := \phi$

for $i = 1, \ldots, n$ do.

for each i-tuple $(b_1, \ldots, b_i) \in A^i$ do

If $(b_1, \ldots, b_i)$ is a path in R, then add (b_1, b_i) to R^*.

TO BE CONTINUED.