What is a partition of a set?

Kun-Mao Chao
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We say that $\Pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ is a partition of a set A if

(1) $\pi_i \neq \phi \quad 1 \le i \le k$;

(2) $\pi_i \cap \pi_j = \phi \quad 1 \le i \neq j \le k$;

(3) $\bigcup_{i=1}^{k} \pi_i = A$.

ex. $A = \{a, b, c, d, e\}$.

✓ $\{\{a, c, d\}, \{b, e\}\}$, ✓ $\{\{a, b\}, \{c, d\}, \{e\}\}$

✗ $\{\{a, b\}, \{c, d\}\}$, ✗ $\{\{a, b, c\}, \{c, d, e\}\}$

Kun-Mao Chao 9/12, 2012

B_n: the number of partitions of a set of n elements.

$S(n,k)$: the number of partitions of a set of n elements with k bins.

$$B_n = \sum_{k=1}^{n} S(n,k)$$

Now we show how to compute B_{10}.

$$B_{10} = \underbrace{S(10,1)}_{=1} + S(10,2) + S(10,3) + \cdots + \underbrace{S(10,9)}_{=\binom{10}{2}=45} + \underbrace{S(10,10)}_{=1}$$

Let $A = \{a_1, a_2, \cdots, a_{10}\}$.

$\{a_1\} \cup \underbrace{\cdots}_{k-1}$ $\qquad$ $\{a_1, \cdots\} \cup \underbrace{\cdots}_{k}$

$$S(10,k) = S(9,k-1) + k\,S(9,k)$$

S: n \ k	1	2	3	4	5	6	7	8	9	10	B_n
1	1										1
2	1	1									2
3	1	3	1								5
4	1	7	6	1							15
5	1	15	25	10	1						52
6	1	31	90	65	15	1					203
7	1	63	301	350	140	21	1				877
8	1	127	966	1701	1050	266	28	1			4140
9	1	255	3025	7770	6951	2646	462	36	1		21147
10	1	511	9330	34105	42525	22827	5880	750	45	1	115975 ← B_{10}

You may compute B_n by another formula: $B_n = \sum_{j=1}^{n} \binom{n-1}{j-1} B_{n-j} = \sum_{j=0}^{n-1} \binom{n-1}{j} B_j$

$|\{a_1, \cdots\}| = j$ $\qquad$ $|\{a_1, \cdots\}| = n-j$

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Kun-Mao Chao
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Let us assume that

(a). Each one is a friend of him/her-self;
(b) If x is a friend of y, y is a friend of x;
(c). A friend of a friend is a friend.

Consider the binary relation of a set A:

$R = \{(x,y): x, y \in A$ and x is a friend of $y\}$.

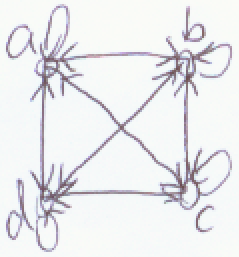
Let $A = \{a, b, c, d, e, f, g, h\}$.

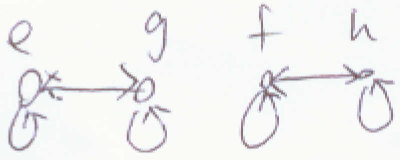
(a) $(a,a) \in R, \dots$ reflexive; (b) $(a,b) \in R \Rightarrow (b,a) \in R$; symmetric
(c) If $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R$. transitive

Equivalence relation

reflexive, symmetric, transitive

$R = \{\overset{[a]}{\{a,b,c,d\}}, \overset{[e]}{\{e,g\}}, \overset{[f]}{\{f,h\}}\}$ Equivalence classes: a partition





Assume that $[a] \neq [e]$, & $[a] \cap [e] \neq \phi$

$x \in [a] \Rightarrow (x,a) \in R$
$x \in [e] \Rightarrow (x,e) \in R$
$\Rightarrow (a,e) \in R$

For any $y \in [a]$, $(y,a) \in R$
$\Rightarrow (y,e) \in R \Rightarrow y \in [e]$
$\Rightarrow [a] \subseteq [e]$

Similarly, $[e] \subseteq [a]$
We have $[a] = [e]$.
A contradiction.

Some Proof Techniques.

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☆ Proof by contradiction

Prove that $\sqrt{2}$ is irrational.

Let $\sqrt{2} = \frac{m}{n}$, where $m \geq 1$ and $n \geq 1$

We assume that m and n are not both even. (For otherwise, ...)

$$\sqrt{2} = \frac{m}{n} \Rightarrow 2n^2 = m^2$$

$$\Rightarrow m^2 \text{ is even}$$

$$\Rightarrow m \text{ is even}$$

Let $m = 2k$

$$2n^2 = (2k)^2 = 4k^2$$

$$n^2 = 2k^2$$

$$\Rightarrow n \text{ is even}$$

A contradiction.

We conclude that $\sqrt{2}$ is irrational.

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※ Nonconstructive proofs

Prove that $\exists$ irrational numbers x and y such that

x^y is rational, i.e., $x^y \in Q$.

Proof. If $\sqrt{2}^{\sqrt{2}} \in Q$, we are done. $(x = \sqrt{2},\ y = \sqrt{2})$

Otherwise, $\sqrt{2}^{\sqrt{2}} \notin Q$. $\left(\underbrace{\sqrt{2}^{\sqrt{2}}}_{\uparrow\, x}\right)^{\sqrt{2}\,\leftarrow y} = \sqrt{2}^{\,2} = 2$.

$(x = \sqrt{2}^{\sqrt{2}},\ y = \sqrt{2})$ #

You may use $\sqrt{3}^{\sqrt{3}}, \cdots$ as well.

The pigeon hole principle.

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<u>Thm.</u> Let n be a positive number. Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either increasing or decreasing.

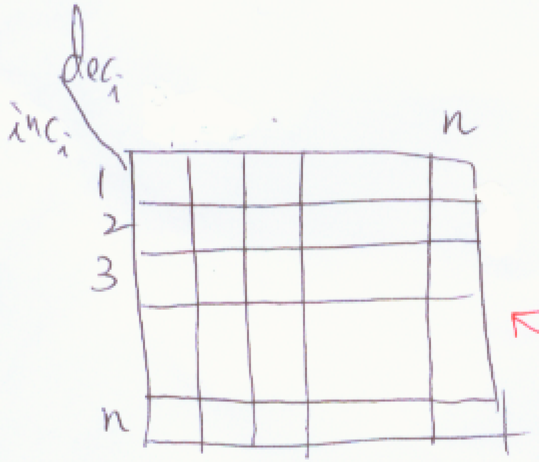
e.g. $(18, \underline{5}, 20, \underline{8}, \underline{19})$

$(a_1, a_2, \ldots, a_{n^2+1})$

inc_i: the length of the longest increasing subsequence starting at a_i.

dec_i: " decreasing "

Assume that $inc_i \le n$ & $dec_i \le n$.



$\leftarrow n^2+1$ numbers

$\leftarrow n^2$ buckets

$\exists\, i<j$ s.t. $inc_i = inc_j$
$dec_i = dec_j$

If $a_i < a_j \Rightarrow inc_i \ge 1 + inc_j$

If $a_i > a_j \Rightarrow dec_i \ge 1 + dec_j$.