We are given two sequences \( A = \langle a_1, a_2, \ldots, a_M \rangle \) and \( B = \langle b_1, b_2, \ldots, b_N \rangle \). An alignment of \( A \) and \( B \) is obtained by introducing dashes into the two sequences such that the lengths of the two resulting sequences are identical and no column contains two dashes. Let \( \Sigma \) denote the input symbol alphabet. A score \( \sigma(a, b) \) is defined for each \((a, b) \in \Sigma \times \Sigma\). The score of an alignment is the sum of \( \sigma \) scores of all columns with no dashes minus the penalties of the gaps.

**Problem 1 (60%)**: In this problem, we employ a simple scoring scheme where each gap symbol is penalized by a nonnegative constant \( \beta \). Let \( S(i, j) \) denote the score of an optimal alignment between \( \langle a_1, a_2, \ldots, a_i \rangle \) and \( \langle b_1, b_2, \ldots, b_j \rangle \). With proper initializations, \( S(i, j) \) can be computed by the following recurrences:

\[
S(i, j) = \max \begin{cases} 
S(i-1, j) - \beta \\
S(i, j-1) - \beta \\
S(i-1, j-1) + \sigma(a_i, b_j)
\end{cases}
\]

(a) (15%): Write down a complete pseudo-code for computing \( S(M, N) \) in \( O(MN) \) time and \( O(MN) \) space. All initializations should be included in the pseudo-code.

(b) (15%): Write down an \( O(MN) \)-time and \( O(M+N) \)-space version of the pseudo-code in (a).

(c) (10%): Describe an \( O(MN) \)-time and \( O(M+N) \)-space approach for delivering an optimal alignment that achieves the score \( S(M, N) \). Justify your answer. No pseudo-code is needed for this subproblem.

(d) (10%): Describe an \( O(MN) \)-time and \( O(M+N) \)-space approach for delivering an optimal local alignment. Justify your answer. No pseudo-code is needed for this subproblem.

(e) (10%): Assume that the maximum length of a gap allowed in an alignment is \( x \). Give a method (as efficient as possible) for computing the score of an optimal alignment. (You may use more than one matrices.)

**Problem 2 (15%)**: In affine gap penalties, a gap of length \( k \) is penalized by \( \alpha + k \times \beta \), where \( \alpha \) and \( \beta \) are both nonnegative constants.

(a) (10%): Give the recurrence relations for computing the score of an optimal (global) alignment between \( A \) and \( B \). Justify your recurrence relations.

(b) (5%): Give the recurrence relations for computing the score of an optimal local alignment between \( A \) and \( B \).

**Problem 3 (15%)**: In restricted affine gap penalties, a gap of length \( k \) is penalized by \( \alpha + f(k) \times \beta \), where \( \alpha \) and \( \beta \) are both nonnegative constants, and \( f(k) = \min\{k, c\} \) for a given positive integer \( c \). Give the recurrence relations for computing the score of an optimal (global) alignment between \( A \) and \( B \). Justify your answer.

**Problem 4 (10%)**: Explain why the approach of dividing the dynamic-programming matrix by both the middle row and middle column is more space efficient for computing \( \Delta \)-points (suboptimal points) than the approach of dividing the matrix by merely the middle row.