Analysis Tools for Data Structures and Algorithms

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Asymptotic Notation

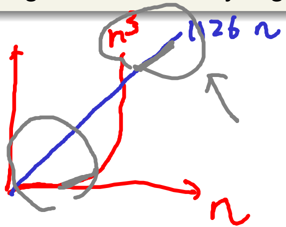
Representing “Rough” by Asymptotic Notation

- goal: rough rather than exact steps
- why rough? constant not matter much

—when input size large

compare two complexity functions $f(n)$ and $g(n)$

growth of functions matters
—when n large, n^3 eventually bigger than $1126n$



rough ⇔ asymptotic behavior

Asymptotic Notations: Rough Upper Bound

big-O: rough upper bound

- $f(n)$ grows slower than (or similar to) $g(n)$: $f(n) = O(g(n))$
  - n grows slower than n^2: $n = O(n^2)$
  - $3n$ grows similar to n: $3n = O(n)$
- asymptotic intuition (rigorous math later):

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \le c$$

$f(n) = O(g(n))$

big-O: arguably the most used "language" for complexity

More Intuitions on Big-O

$$f(n) = O(g(n)) \Leftarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c \quad \text{(not rigorously, yet)}$$

1126

$\sqrt{n}$

n

$n^2 = O(n^2)$

$0.1n^2$

$3n^2$

- "$= O(\cdot)$" more like "$\in$"
  - $n = O(n)$
  - $n = O(10n)$
  - $n = O(0.3n)$
  - $n = O(n^5)$
- "$= O(\cdot)$" also like "$\leq$"
  - $n = O(n^2)$
  - $n^2 = O(n^{2.5})$
  - $n = O(n^{2.5})$
- $1126n = O(n)$: coefficient not matter
- $n + \cancel{\sqrt{n}} + \cancel{\log n} = O(n)$: lower-order term not matter

"exact"

intuitions (properties) to be proved later

Formal Definition of Big-O

$\lim \frac{f}{g} \le c$

$f(n) > 0$

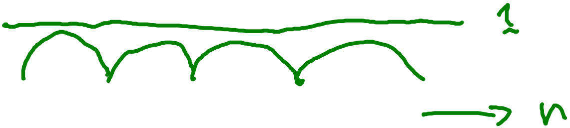
$g(n) > 0$

Consider positive functions $f(n)$ and $g(n)$,

$f(n) = O(g(n))$, iff exist c, n_0 such that $f(n) \le c \cdot g(n)$ for all $n \ge n_0$

- covers the lim intuition if limit exists
- covers other situations without "limit"
  e.g. $|\sin(n)| = O(1)$

next: prove that lim intuition $\Rightarrow$ formal definition



lim Intuition ⇒ Formal Definition

$n \geq n_0$
$f \leq c' g$
(c', n_0')

For positive functions f and g, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c$, then $f(n) = O(g(n))$.

- with definition of limit, there exists ϵ, n_0 such that for all $n \geq n_0$, $|\frac{f(n)}{g(n)} - c| < \epsilon$.

  n_0' $\qquad$ c'

- That is, for all $n \geq n_0$, $\frac{f(n)}{g(n)} < c + \epsilon$.
- Let $c' = c + \epsilon$, $n_0' = n_0$, big-O definition satisfied with (c', n_0'). QED.

important to not just have intuition (building),
but know definition (building block)

More on Asymptotic Notations

Asymptotic Notations: Definitions

- $f(n)$ grows slower than or similar to $g(n)$: ("$\leq$")

  $f(n) = O(g(n))$, iff exist c, n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

- $f(n)$ grows faster than or similar to $g(n)$: ("$\geq$") $c > 0$

  $f(n) = \Omega(g(n))$, iff exist c, n_0 such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$

  big-Omega

- $f(n)$ grows similar to $g(n)$: ("$\approx$")

  $f(n) = \Theta(g(n))$, iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

let's see how to use them

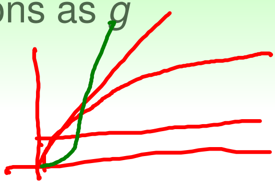
Ω

$8 - \epsilon < \frac{f}{g} < 8 + \epsilon$

$\lim \frac{f}{g} \to 8$ 0

The Seven Functions as g

$f(n) = O(1)$



$g(n) = ?$

- 1: constant
- $\log n$: logarithmic (does base matter?)
- n: linear
- $n \log n$
- n^2: square
- n^3: cubic
- 2^n: exponential (does base matter?)

$f(n) = O(\log n)$

$\log_2 n = \frac{\log_{10} n}{\log_{10} 2}$

will often encounter them in future classes

Analysis of Sequential Search

Sequential Search

```
for i ← 0 to n − 1 do
    if list[i] == num
        return i
    end if
end for
return −1
```

- best case (e.g. *num* at 0): time $\Theta(1)$
- worst case (e.g. *num* at last or not found): time $\Theta(n)$

often just say $O(n)$-algorithm (linear complexity)

Analysis of Binary Search

Binary Search

left $\leftarrow 0$, *right* $\leftarrow n - 1$
while *left* $\leq$ *right* **do**
  mid $\leftarrow$ floor$((left + right)/2)$
  if *list*[*mid*] $>$ *num*
    left $\leftarrow$ *mid* $+ 1$
  else if *list*[*mid*] $<$ *num*
    right $\leftarrow$ *mid* $- 1$
  else
    return *mid*
  end if
end while
return -1

- best case (e.g. *num* at *mid*): time $\Theta(1)$
- worst case (e.g. *num* not found): because range (*right* $-$ *left*) halved in each WHILE, needs time $\Theta(\log n)$ iterations to decrease range to 0

$16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$

$\log_2 16$

often just say $O(\log n)$-algorithm (logarithmic complexity)

Sequential and Binary Search

- Input: any integer array *list* with size n, an integer *num*
- Output: if *num* not within *list*, -1; otherwise, $+1126$

DIRECT-SEQ-SEARCH (*list*, *n*, *num*)

```
for i ← 0 to n − 1 do
    if list[i] == num
        return +1126
    end if
end for
return −1
```

SORT-AND-BIN-SEARCH (*list*, *n*, *num*)

```
SEL-SORT(list, n)
return
BIN-SEARCH(list, n, num) ≥ 0? + 1126 : −1
```

- DIRECT-SEQ-SEARCH: $O(n)$ time ✓
- SORT-AND-BIN-SEARCH: $O(n^2)$ time for SEL-SORT and $O(\log n)$ time for BIN-SEARCH ✓ ✓

next: operations for "combining" asymptotic complexity

Properties of Asymptotic Notations

Some Properties of Big-O I

Theorem (封閉律)

if $f_1(n) = O(g_2(n))$, $f_2(n) = O(g_2(n))$ then $f_1(n) + f_2(n) = O(g_2(n))$

- When $n \geq n_1$, $f_1(n) \leq c_1 g_2(n)$
- When $n \geq n_2$, $f_2(n) \leq c_2 g_2(n)$
- So, when $n \geq \max(n_1, n_2)$, $f_1(n) + f_2(n) \leq (c_1 + c_2) g_2(n)$

n_0' c' QED

Theorem (遞移律)

if $f_1(n) = O(g_1(n))$, $g_1(n) = O(g_2(n))$ then $f_1(n) = O(g_2(n))$

- When $n \geq n_1$, $f_1(n) \leq c_1 g_1(n)$
- When $n \geq n_2$, $g_1(n) \leq c_2 g_2(n)$
- So, when $n \geq \max(n_1, n_2)$, $f_1(n) \leq c_1 c_2 g_2(n)$

n_0' c'

Some Properties of Big-O II

sel-sort is $O(n^2)$ bin-search is $O(\log n)$

f_2 g_1 f_1 g_2

Theorem (併吞律)

if $f_1(n) = O(g_1(n))$, $f_2(n) = O(g_2(n))$ *and* $g_1(n) = O(g_2(n))$ *then* $f_1(n) + f_2(n) = O(g_2(n))$

Proof: use two theorems above.

$\log n$ is $O(n^2)$

Theorem

If $f(n) = a_m n^m + \cdots + a_1 n + a_0$, *then* $f(n) = O(n^m)$

Proof: use the theorem above.

similar proof for Ω and Θ

$\underbrace{f_1 + f_2}_{\text{sel-sort-bin-search}} = O(n^2)$

Some More on Big-O

1 "$\leq$" 10

$\leq$ 1126

RECURSIVE-BIN-SEARCH is $O(\log n)$ time and $O(\log n)$ space

- by 遞移律, time also $O(n)$
- time also $O(n \log n)$ $\leq$ 5566
- time also $O(n^2)$
- also $O(2^n)$ $\leq$ ...
- ...

prefer the tightest Big-O!

Practical Complexity

some input sizes are time-wise infeasible for some algorithms

when 1-billion-steps-per-second

n	n	$n \log_2 n$	n^2	n^3	n^4	n^{10}	2^n
10	$0.01\mu s$	$0.03\mu s$	$0.1\mu s$	$1\mu s$	$10\mu s$	$10s$	$1\mu s$
20	$0.02\mu s$	$0.09\mu s$	$0.4\mu s$	$8\mu s$	$160\mu s$	$2.84h$	$1ms$
30	$0.03\mu s$	$0.15\mu s$	$0.9\mu s$	$27\mu s$	$810\mu s$	$6.83d$	$1s$
40	$0.04\mu s$	$0.21\mu s$	$1.6\mu s$	$64\mu s$	$2.56ms$	$121d$	$18m$
50	$0.05\mu s$	$0.28\mu s$	$2.5\mu s$	$125\mu s$	$6.25ms$	$3.1y$	$13d$
100	$0.10\mu s$	$0.66\mu s$	$10\mu s$	$1ms$	$100ms$	$3171y$	$4 \cdot 10^{13}y$
10^3	$1\mu s$	$9.96\mu s$	$1ms$	$1s$	$16.67m$	$3 \cdot 10^{13}y$	$3 \cdot 10^{284}y$
10^4	$10\mu s$	$130\mu s$	$100ms$	$1000s$	$115.7d$	$3 \cdot 10^{23}y$	
10^5	$100\mu s$	$1.66ms$	$10s$	$11.57d$	$3171y$	$3 \cdot 10^{33}y$	
10^6	$1ms$	$19.92ms$	$16.67m$	$32y$	$3 \cdot 10^7y$	$3 \cdot 10^{43}y$	

note: similar for space complexity,
e.g. store an N by N double matrix when $N = 50000$?

100000