**Homework #2 Selected Solution**

## 2.1 Asymptotic Complexity

In this problem, you can use any theorems in the textbook and any theorems on the class slides as the foundation of your proof. You **cannot** use any other theorems unless you prove them first.

(1) (10%) Do Exercise 1(b) on page 41 of the textbook (Subsec. 1.5.3).

**Proof.** Let $n_0 = 0$ and $c = 1$. For $n \ge n_0$, we see that

$$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1 \\ &\le n \cdot n \cdot n \cdot \cdots \cdot n \\ &= c \cdot n^n \end{aligned}$$

Thus, by the definition in the textbook, $n! = O(n^n)$.

(2) (10%) Do Exercise 1(f) on page 41 of the textbook (Subsec. 1.5.3).

**Proof.** We will take $c_1 = 1$, $c_2 = 2$, $n_0 = 3$ and try to use the definition to prove the results.

For $n \ge n_0 = 3$, $n$ must be $\ge 1$. Then, $2^n \ge 2 \ge 0$. Thus,

$$n^{2^n} + 6 \cdot 2^n \ge n^{2^n} = c_1 \cdot n^{2^n}.$$

Now, for $n \ge n_0 = 3$, we shall prove that the property $2^n \ge 2n$ first. The proof can be done with a mathematical induction. When $n = 3$ we see that the property is true by $2^n = 8 \ge 6 = 2n$. Assume that the property is true for $n = k$. Then, for $n = k+1 \ge 3$, we first know that $k \ge 2 \ge 1$. That is, $2^k \ge 2$. Thus,

$$2^n = 2^{k+1} = 2^k + 2^k \ge 2k + 2 = 2(k+1).$$

Note that we used the assumed $2^k \ge 2k$ in the derivation. By mathematical induction, $2^n \ge 2n$ is true for all $n \ge n_0 = 3$.

Then, since $\log_2 n \ge \log_2 3 \ge 1$ for $n \ge n_0 = 3$, and $n \ge n_0 = 3 \ge \log_2 6$,

$$2^n \log_2 n \ge 2^n \ge 2n \ge n + \log_2 6.$$

Thus,

$$n^{2^n} \ge 6 \cdot 2^n.$$

That is,

$$n^{2^n} + 6 \cdot 2^n \le 2 \cdot n^{2^n} = c_2 n^{2^n}.$$

(3) (10%) Do Exercise 2(c) on page 41 of the textbook (Subsec. 1.5.3).

Assume that the statement is true, there exists $c_2$ and $n_0$ such that

$$c_2 n^2 \le n^2 / \log n$$

for all $n \ge n_0$. That is,

$$\log n \le \frac{1}{c_2}$$

Take $n = \max(n_0, 10^{\frac{1}{c_2}+1})$. We see that $n \ge n_0$ but

$$\log n \ge \frac{1}{c_2} + 1 > \frac{1}{c_2},$$

which is a contradiction. Thus, the statement is not true.

(4) (10%) Do Exercise 2(e) on page 41 of the textbook (Subsec. 1.5.3).

Assume that the statement is true, there exists $c$ and $n_0$ such that

$$3^n \le c \cdot 2^n$$

for all $n \ge n_0$. That is,

$$n(\log_2 3 - 1) \le \log_2 c$$

Take $n = \max(n_0, \frac{\log_2 c}{\log_2 3 - 1} + 1)$. We see that $n \ge n_0$ but

$$n(\log_2 3 - 1) \ge \log_2 c + \log_2 3 - 1 > \log_2 c,$$

which is a contradiction. Thus, the statement is not true.