# Visibility Representations of Four-Connected Plane Graphs with Near Optimal Heights

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**Abstract.** A *visibility representation* of a graph $G$ is to represent the nodes of $G$ with non-overlapping horizontal line segments such that the line segments representing any two distinct adjacent nodes are vertically visible to each other. If $G$ is a plane graph, i.e., a planar graph equipped with a planar embedding, a visibility representation of $G$ has the additional requirement of reflecting the given planar embedding of $G$. For the case that $G$ is an $n$-node four-connected plane graph, we give an $O(n)$-time algorithm to produce a visibility representation of $G$ with height at most $\lceil \frac{n}{2} \rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$. To ensure that the first-order term of the upper bound is optimal, we also show an $n$-node four-connected plane graph $G$, for infinite number of $n$, whose visibility representations require heights at least $\frac{n}{2}$.

## 1 Introduction

Unless clearly specified otherwise, all graphs in the present article are simple, i.e., having no self-loops and multiple edges. A *visibility representation* of a planar graph represents the nodes of the graph by non-overlapping horizontal line segments such that, for any nodes $u$ and $v$ adjacent in the graph, the line segments representing $u$ and $v$ are vertically visible to each other. Observe that if $G_1$ is a subgraph of $G_2$ on the save node set, then any visibility representation of $G_2$ is also a visibility representation of $G_1$. Therefore, we may assume without loss of generality that the input graph is maximally planar. Let $G$ be an $n$-node plane triangulation, i.e., a maximally planar graph equipped with a planar embedding. A visibility representation of $G$ has an additional requirement of reflecting the given planar embedding of $G$. Figure 1(b), for instance, is a visibility representation of the four-connected plane graph shown in Fig. 1(a). Under the conventional restriction of placing the endpoints of horizontal line segments on the integral grid points, any visibility representation of $G$ requires width no more than $3n-7$ and height no more than $n-1$. Otten and van Wijk [7] gave the first known algorithm for constructing a visibility representation for any $G$. Rosenstiehl and Tarjan [8] and Tamassia

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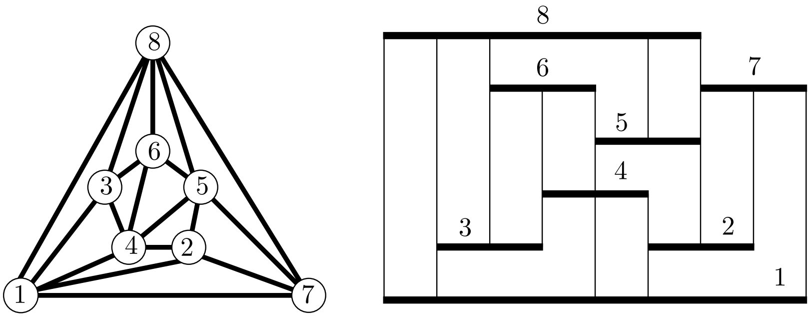



(a) (b)

**Fig. 1.** (a) A four-connected plane triangulation $G$. (b) A visibility representation of $G$.

and Tollis [9] independently gave algorithms to compute a visibility representation of $G$ with height at most $2n-5$. Their work initiated a decade of competition on minimizing the width and height of the output visibility representation. All these algorithms run in linear time. In particular, the results of Fan, Lin, Lu, and Yen [2] and Zhang and He [16] are optimal in that the upper bounds differ from the best known lower bounds by very small constants.

The present article focuses on four-connected plane $G$. The $O(n)$-time algorithm of Kant and He [5] provides the optimal upper bound $n-1$ on the width. The best previously known upper bound on the height, ensured by the $O(n)$-time algorithm of Zhang and He [12], is $\lceil \frac{3n}{4} \rceil$. In the present article, we obtain the following result with an improved upper bound on the required height.

**Theorem 1.** *For any $n$-node four-connected plane graph $G$, it takes $O(n)$ time to construct a visibility representation of $G$ with height at most $\lceil \frac{n}{2} \rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$.*

Table 1 compares our upper bound with previous results. All algorithms shown in Table 1 run in $O(n)$ time. Our algorithm follows the approach of Zhang and He [10, 15–17], originating from Rosenstiehl and Tarjan [8] and Tamassia and Tollis [9], that reduces the problem of computing a visibility representation for $G$ with small height to finding an appropriate $st$-ordering of $G$. To find such an $st$-ordering of $G$, we resort to three linear-time obtainable node orderings:

- four-canonical orderings of four-connected plane graphs (Kant and He [5]),
- consistent orderings of ladder graphs (Zhang and He [15–17]), and
- post-orderings of canonical ordering spanning trees (He, Kao, and Lu [3]).

Our result is near optimal in that we can construct an $n$-node four-connected plane graph, for infinite number of $n$, whose visibility representations require heights at least $\lceil \frac{n}{2} \rceil$. That is, the first-order term of our upper bound is optimal.

The remainder of the paper is organized as follows. Section 2 gives the preliminaries. Section 3 describes and analyzes our algorithm. Section 4 ensures that the first-order term of our upper bound on height is optimal. Section 5 concludes the paper.

**Table 1.** Previous upper bounds and our result for any $n$-node plane graph $G$

| | general $G$ | | four-connected $G$ | |
|---|---|---|---|---|
| | width | height | width | height |
| Otten and van Wijk [7] | $3n-7$ | $n-1$ | | |
| Rosenstiehl and Tarjan [8], Tamassia and Tollis [9] | $2n-5$ | | | |
| Kant [4] | $\lfloor \frac{3n-6}{2} \rfloor$ | | | |
| Kant and He [5] | | | $n-1$ | |
| Lin, Lu, and Sun [6] | $\lfloor \frac{22n-24}{15} \rfloor$ | | | |
| Zhang and He [10] | | $\lceil \frac{15n}{16} \rceil$ | | |
| Zhang and He [14] | | $\lfloor \frac{5n}{6} \rfloor$ | | |
| Zhang and He [11, 13] | $\lfloor \frac{13n-24}{9} \rfloor$ | | | |
| Zhang and He [12] | | | | $\lceil \frac{3n}{4} \rceil$ |
| Zhang and He [15, 17] | $\frac{4n}{3} + 2\lceil \sqrt{n} \rceil$ | $\frac{2n}{3} + 2\lceil \sqrt{\frac{n}{2}} \rceil$ | | |
| Zhang and He [16] | | $\frac{2n}{3} + O(1)$ | | |
| Fan, Lin, Lu, and Yen [2] | $\lfloor \frac{4n}{3} \rfloor - 2$ | | | |
| This paper | | | | $\lceil \frac{n}{2} \rceil + 2\left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$ |

## 2 Preliminaries

### 2.1 Ordering and $st$-Ordering

Let $G$ be an $n$-node plane graph. An *ordering* of $G$ is a one-to-one mapping $\sigma$ from the nodes of $G$ to $\{1, 2, \ldots, n\}$. A path of $G$ is $\sigma$*-increasing* if $\sigma(u) < \sigma(v)$ holds for any nodes $u$ and $v$ such that $u$ precedes $v$ in the path. Let $length(G, \sigma)$ denote the maximum of the lengths of all $\sigma$-increasing paths in $G$. For instance, if $G$ and $\sigma$ are as shown in Fig. 1(a), then one can verify that $(1, 2, 5, 6, 8)$ is a $\sigma$-increasing path with maximum length. Therefore, $length(G, \sigma) = 4$.

Let $s$ and $t$ be two distinct external nodes of $G$. An $st$*-ordering* [1] of $G$ is an ordering $\sigma$ of $G$ such that

- $\sigma(s) = 1$, $\sigma(t) = n$, and
- each node $v$ of $G$ other than $s$ and $t$ has neighbors $u$ and $w$ in $G$ with $\sigma(u) < \sigma(v) < \sigma(w)$.

An example is shown in Fig. 1(a): the node labels form an $st$-ordering for the graph.

The following lemma reduces the problem of minimizing the height of visibility representation of $G$ to that of finding an $st$-ordering $\sigma$ of $G$ with minimum $length(G, \sigma)$.

**Lemma 1 (See [2, 8–10, 15, 17]).** *If $G$ admits an $st$-ordering $\sigma$ for two distinct external nodes $s$ and $t$ of $G$, then it takes $O(n)$ time to obtain a visibility representation of $G$ with height exactly $length(G, \sigma)$.*

For instance, if $G$ and $\sigma$ are as shown in Fig. 1(a), then a visibility representation for $G$ with height at most $length(G, \sigma) = 4$, as shown in Fig. 1(b), can be found in linear time.

### 2.2 Four-Canonical Ordering

Let $G$ be an $n$-node four-connected plane triangulation. Let $v_1$, $v_2$, and $v_n$ be the external nodes of $G$ in counterclockwise order. Since $G$ is a four-connected plane triangulation, $G$ has exactly one internal node adjacent to both $v_2$ and $v_n$. Let $v_{n-1}$ be the internal node adjacent to $v_2$ and $v_n$ in $G$. A *four-canonical ordering* [5] of $G$ is an ordering $\phi$ in $G$ such that

- $\phi(v_1) = 1$, $\phi(v_2) = 2$, $\phi(v_{n-1}) = n - 1$, $\phi(v_n) = n$, and
- each node $v$ of $G$ other than $v_1$, $v_2$, $v_{n-1}$ and $v_n$ has neighbors $u$, $u'$, $w$ and $w'$ in $G$ with $\phi(u') < \phi(u) < \phi(v) < \phi(w) < \phi(w')$.

An example is shown in Fig. 2(a): the node labels form a four-canonical ordering of the four-connected plane triangulation.

**Lemma 2 (Kant and He [5]).** *It takes $O(n)$ time to compute a four-canonical ordering for any $n$-node $G$.*

### 2.3 Consistent Ordering of Ladder Graph

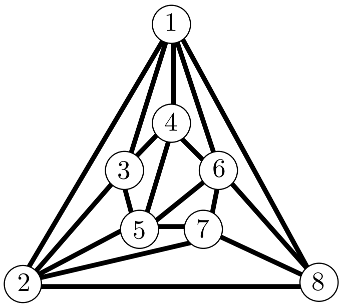
Let $L$ be an $\lceil \frac{n}{2} \rceil$-node path. Let $R$ be an $\lfloor \frac{n}{2} \rfloor$-node path. Let $X$ consist of edges with one endpoint in $L$ and the other endpoint in $R$. Let $(L, R, X)$ denote the $n$-node graph $L \cup R \cup X$. We say that $(L, R, X)$ is a *ladder graph* [15, 17] if $L \cup R \cup X$ is outerplanar. A ladder graph is shown in Fig. 3(a).

An ordering $\sigma$ of ladder graph $(L, R, X)$ is *consistent* [15, 17] with respect to an outerplanar embedding $\mathcal{E}$ of $(L, R, X)$ if $L$ (respectively, $R$) forms a $\sigma$-increasing path in clockwise (respectively, counterclockwise) order according to $\mathcal{E}$. See Fig. 3(a) for an example: The node labels form a consistent ordering of the ladder graph with respect to the displayed outerplanar embedding.

**Lemma 3 (He and Zhang [15, 17]).** *Let $(L, R, X)$ be an $n$-node ladder graph. It takes $O(n)$ time to compute a consistent ordering $\sigma$ of $(L, R, X)$ with respect to any given outerplanar embedding of $(L, R, X)$ such that $length((L, R, X), \sigma) \leq \lceil \frac{n}{2} \rceil + 2 \lceil \sqrt{\frac{n}{2}} \rceil - 1$.*

For technical reason, we need a consistent ordering with additional properties, as stated in the next lemma, which is also illustrated by Fig. 3(a).

**Lemma 4.** *Let $(L, R, X)$ be an $n$-node ladder graph. It takes $O(n)$ time to compute a consistent ordering $\sigma$ of $(L, R, X)$ with respect to any given outerplanar embedding $\mathcal{E}$ of $(L, R, X)$ such that*



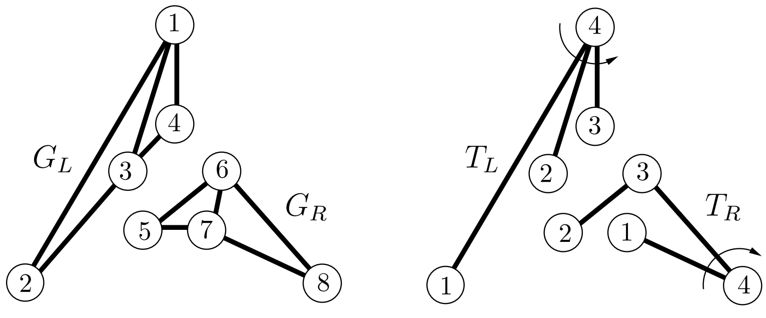




**Fig. 2.** (a) A four-canonical ordering $\phi$ of the four-connected plane triangulation $G$. (b) $G_L$ is the subgraph induced by the nodes $v$ with $1 \leq \phi(v) \leq 4$ and $G_R$ is the subgraph induced by the nodes $v$ with $5 \leq \phi(v) \leq 8$. (c) The counterclockwise post-ordering $\psi_L$ of $T_L$ and the clockwise post-ordering $\psi_R$ of $T_R$.

– $\sigma(\ell_1) = 1$, $\sigma(r_1) = 2$, *and*
– *length*$((L, R, X), \sigma) \leq \lceil \frac{n}{2} \rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$,

*where* $\ell_1$ (*respectively,* $r_1$) *is the first* (*respectively, last*) *node of* $L$ (*respectively,* $R$) *in clockwise order around the external boundary of* $(L, R, X)$ *with respect to* $\mathcal{E}$.

*Proof.* Let $L' = L \setminus \{\ell_1\}$. Let $R' = R \setminus \{r_1\}$. Let $X' = X \setminus \{\ell_1, r_1\}$. Clearly, $(L', R', X')$ is a ladder graph of $n - 2$ nodes. Let $\sigma'$ be the consistent ordering of $(L', R', X')$ with respect to $\mathcal{E}$ ensured by Lemma 3. We have

$$\textit{length}((L', R', X'), \sigma') \leq \left\lceil \frac{n}{2} \right\rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil - 2.$$

Let $\sigma$ be the ordering of $(L, R, X)$ such that

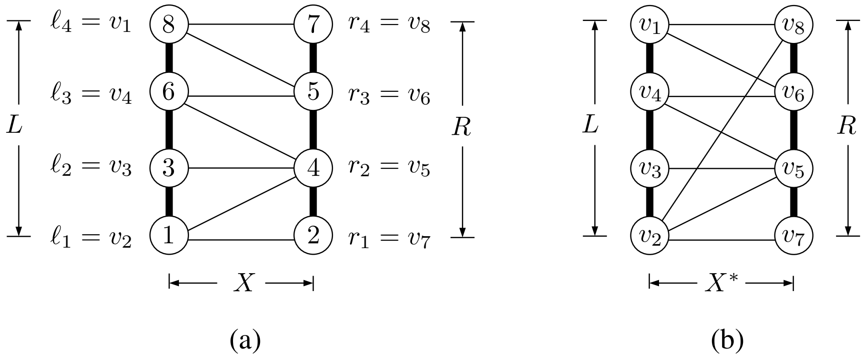



**Fig. 3.** (a) A consistent ordering of a ladder graph $(L, R, X)$ with respect to the displayed outer-planar embedding. (b) $H^* = L \cup R \cup X^*$, where $X^* = X \cup \{(v_2, v_8)\}$.

- $\sigma(\ell_1) = 1$, $\sigma(r_1) = 2$, and
- $\sigma(u) = \sigma'(u) + 2$ holds for each node $u$ other than $\ell_1$ and $r_1$.

One can easily verify that the lemma holds. □

# 3 Our Algorithm

Let $G$ be the input $n$-node four-connected plane triangulation. According to Lemma 1, it suffices to describe our algorithm for computing an $st$-ordering $\sigma$ for $G$ in the following four steps.

## 3.1 Step 1

Let $\phi$ be a four-canonical ordering of $G$ ensured by Lemma 2.

- Let $G_L$ be the subgraph of $G$ induced by the nodes $v$ with $1 \le \phi(v) \le \lceil \frac{n}{2} \rceil$.
- Let $G_R$ be the subgraph of $G$ induced by the nodes $v$ with $\lceil \frac{n}{2} \rceil < \phi(v) \le n$.

Figure 2(b) illustrates this step, which runs in $O(n)$ time. Observe that each edge of $G$ not in $G_L \cup G_R$ has one endpoint on the external boundary of $G_L$ and the other endpoint on the external boundary of $G_R$.

## 3.2 Step 2

For each $i = 1, 2, \ldots, n$, let $v_i$ denote the node of $G$ with $\phi(v_i) = i$. It follows from the definition of $\phi$ that $v_1$, $v_2$, and $v_n$ are the external nodes of $G$.

- For each $i = 2, 3, \ldots, \lceil \frac{n}{2} \rceil$, let $\pi(i)$ be the index $j$ with $j < i$ such that $v_j$ is the first neighbor of $v_i$ in $G_L$ in counterclockwise order around $v_i$. Let $T_L$ be the spanning tree of $G_L$ rooted at $v_1$ such that each $v_{\pi(i)}$ is the parent of $v_i$ in $T_L$. Let $\psi_L$ be the counterclockwise post-ordering of $T_L$.

– For each $i = \lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \ldots, n-1$, let $\pi(i)$ be the index $j$ with $j > i$ such that $v_j$ is the first neighbor of $v_i$ in $G_R$ in clockwise order around $v_i$. Let $T_R$ be the spanning tree of $G_R$ rooted at $v_n$ such that each $v_{\pi(i)}$ is the parent of $v_i$ in $T_R$. Let $\psi_R$ be the clockwise post-ordering of $T_R$.

Figure 2(c) illustrates this step, which runs in $O(n)$ time. As a matter of fact, $T_L$ is the canonical ordering spanning tree of $G_L$ with respect to $\phi$, as defined by He, Kao, and Lu [3].

**Lemma 5.** $\psi_L(v_2) = 1$, $\psi_L(v_1) = \lceil \frac{n}{2} \rceil$, $\psi_R(v_{n-1}) = 1$, *and* $\psi_R(v_n) = \lfloor \frac{n}{2} \rfloor$.

*Proof.* Since $\phi$ is a four-canonical ordering of $G$, if $(v_2, v_i)$ with $i \geq 3$ is an edge of $G_L$, then $v_i$ has to have a neighbor $v_k$ with $2 \neq k < i$ in $G_L$. Observe that $v_2$ is the node immediately succeeding $v_1$ in counterclockwise order around the external boundary of $G_L$. One can verify that $v_2$ cannot be the first neighbor of $v_i$ in $G_L$ in counterclockwise order around $v_i$. That is, we have $\pi(i) \neq 2$. Since $v_2$ cannot be the parent of $v_i$ in $T_L$, $v_2$ has to be a leaf of $T_L$. By the relative position between $v_2$ and $v_1$, it is clear that $v_2$ is the first node in the counterclockwise post-ordering of $T_L$, i.e., $\psi_L(v_2) = 1$.

One can prove $\psi_R(v_{n-1}) = 1$ analogously, where $v_n$ (respectively, $v_{n-1}$, $\psi_R$, $T_R$, and $G_R$) plays the role of $v_1$ (respectively, $v_2$, $\psi_L$, $T_L$, and $G_L$). Since $v_1$ is the root of $T_L$ and $\psi_L$ is a post-ordering of $T_L$, we have $\psi_L(v_1) = \lceil \frac{n}{2} \rceil$. Since $v_n$ is the root of $T_R$ and $\psi_R$ is a post-ordering of $T_R$, we have $\psi_R(v_n) = \lfloor \frac{n}{2} \rfloor$. □

### 3.3 Step 3

Let $L$, $R$, and $X$ be defined as follows.

– Let $L$ be the path $\left(\ell_1, \ell_2, \ldots, \ell_{\lceil n/2 \rceil}\right)$, where $\ell_i$ is the node of $G_L$ with $\psi_L(\ell_i) = i$.
– Let $R$ be the path $\left(r_1, r_2, \ldots, r_{\lfloor n/2 \rfloor}\right)$, where $r_i$ is the node of $G_R$ with $\psi_R(r_i) = i$.
– Let $X = X^* \setminus \{(v_2, v_n)\}$, where $X^*$ consists of the edges of $G$ with one endpoint in $L$ and the other endpoint in $R$.

Figure 3(a) illustrates Lemma 5 and this step, which runs in $O(n)$ time. Figure 3(b) shows the corresponding $L \cup R \cup X^*$.

**Lemma 6.** $(L, R, X)$ *is an* $n$*-node ladder graph.*

*Proof.* Consider any edge $(\ell_i, r_j)$ of $X$. By definition of $\phi$, $\ell_i$ has to be on the external boundary of $G_L$ and $r_j$ has to be on the external boundary of $G_R$. By definition of $T_L$, $\ell_i$ is either a leaf of $T_L$ or on the rightmost path of $T_L$. By definition of $\psi_L$, if $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_p}$ with $i_1 = 1$ are the nodes on the external boundary of $G_L$ in counterclockwise order, then $i_1 < i_2 < \cdots < i_p$. Similarly, by definition of $T_R$, $r_j$ is either a leaf of $T_R$ or on the leftmost path of $T_R$. By definition of $\psi_R$, if $r_{j_1}, r_{j_2}, \ldots, r_{j_q}$ with $j_1 = 1$ are the nodes on the external boundary of $G_R$ in clockwise order, then $j_1 < j_2 < \cdots < j_q$. Since $G$ is a plane graph and the edges of $X$ do not cross one another in $G$, the edges of $X$ do not cross one another in $(L, R, X)$. Therefore, $(L, R, X)$ is outerplanar. □

### 3.4 Step 4

Let $H = (L, R, X)$. Lemma 6 ensures that $H$ is an $n$-node ladder graph. Consider the outerplanar embedding $\mathcal{E}$ of $H$ such that

$$\ell_1, \ell_2, \ldots, \ell_{\lceil n/2 \rceil}, r_{\lfloor n/2 \rfloor}, r_{\lfloor n/2 \rfloor - 1}, \ldots, r_1$$

are the nodes in clockwise order around the external boundary of $H$. Let the output $\sigma$ of our algorithm be the consistent ordering of $H$ with respect to $\mathcal{E}$ ensured by Lemma 4. Figure 3(a) illustrates this step, which also runs in $O(n)$ time.

**Lemma 7.** *The $O(n)$-time obtainable $\sigma$ is an $st$-ordering of $G$ with $\sigma(v_2) = 1$ and $\max(\sigma(v_1), \sigma(v_n)) = n$.*

*Proof.* We first show that $\psi_L$ is an $st$-ordering of $G_L$. Let $i$ be an index with $2 \le i < \lceil \frac{n}{2} \rceil$. Let $k$ be the index such that $\ell_k$ is the parent of $\ell_i$ in $T_L$. Since $\psi_L$ is a post-ordering of $T_L$, we know that $\ell_k$ is a neighbor of $\ell_i$ in $G_L$ with $i < k$. Let $j$ be the index such that $\ell_j$ is the neighbor of $\ell_i$ in $G_L$ immediately succeeding $\ell_k$ in counterclockwise order around $\ell_i$. Recall that $\ell_k$ is the first neighbor of $\ell_i$ in $G_L$ with $\phi(\ell_k) < \phi(\ell_i)$ in counterclockwise order around $\ell_i$. Since $\phi$ is a four-canonical ordering of $G$, we also have $\phi(\ell_j) < \phi(\ell_i)$. Since $\psi_L$ is the counterclockwise post-ordering of $T_L$, we have $\psi(\ell_j) < \psi(\ell_i)$, i.e., $j < i$. Since $\ell_j$ and $\ell_k$ are two neighbors of $\ell_i$ in $G_L$ with $j < i < k$, we know that $\psi_L$ is an $st$-ordering of $G_L$. It can be proved analogously that $\psi_R$ is an $st$-ordering of $G_R$.

Since $\sigma$ is a consistent ordering of $H$ with respect to $\mathcal{E}$, we know that $1 \le i < j \le \lceil \frac{n}{2} \rceil$ implies $\sigma(\ell_i) < \sigma(\ell_j)$ and $1 \le i < j \le \lfloor \frac{n}{2} \rfloor$ implies $\sigma(r_i) < \sigma(r_j)$. We have the following observations.

- Since $\psi_L$ is an $st$-ordering of $G_L$, for each $i = 1, \ldots, \lceil \frac{n}{2} \rceil - 1$, $\ell_i$ has a neighbor $\ell_k$ in $G_L$ with $i < k$. Since $G_L$ is a subgraph of $G$, $\ell_k$ is a neighbor of $\ell_i$ in $G$ with $\sigma(\ell_i) < \sigma(\ell_k)$.
- Since $\psi_L$ is an $st$-ordering of $G_L$, for each $i = 2, \ldots, \lceil \frac{n}{2} \rceil$, $\ell_i$ has a neighbor $\ell_j$ in $G_L$ with $j < i$. Since $G_L$ is a subgraph of $G$, we know that $\ell_j$ is a neighbor of $\ell_i$ in $G$ with $\sigma(\ell_j) < \sigma(\ell_i)$.
- Since $\psi_R$ is an $st$-ordering of $G_R$, for each $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, $r_i$ has a neighbor $r_k$ in $G_R$ with $i < k$. Since $G_R$ is a subgraph of $G$, we know that $r_k$ is a neighbor of $r_i$ in $G$ with $\sigma(r_i) < \sigma(r_k)$.
- Since $\psi_R$ is an $st$-ordering of $G_R$, for each $i = 2, \ldots, \lfloor \frac{n}{2} \rfloor$, $r_i$ has a neighbor $r_j$ in $G_R$ with $j < i$. Since $G_R$ is a subgraph of $G$, we know that $r_j$ is a neighbor of $r_i$ in $G$ with $\sigma(r_j) < \sigma(r_i)$.

According to the above observations, it suffices to ensure that edges $(\ell_1, r_1)$ and $(\ell_{\lceil n/2 \rceil}, r_{\lfloor n/2 \rfloor})$ belong to $G$. By Lemma 5, $\ell_1 = v_2$, $r_1 = v_{n-1}$, $\ell_{\lceil n/2 \rceil} = v_1$, and $r_{\lfloor n/2 \rfloor} = v_n$. Since $v_1$ and $v_n$ are external nodes of the plane triangulation $G$, we know that $(\ell_{\lceil n/2 \rceil}, r_{\lfloor n/2 \rfloor}) = (v_1, v_n)$ is an edge of $G$. By definition of four-canonical ordering $\phi$, we know that $v_{n-1}$ is adjacent to $v_2$. Therefore, $(\ell_1, r_1) = (v_2, v_{n-1})$ is an edge of $G$. $\square$

Figure 1(a) shows the resulting $st$-ordering $\sigma$ of $G$ computed by our algorithm.

### 3.5 Proving Theorem 1

*Proof.* Note that $v_1$, $v_2$, and $v_n$ are the external nodes of $G$. By Lemmas 1 and 7, it suffices to ensure

$$length(G,\sigma) \leq \left\lceil \frac{n}{2} \right\rceil + 2\left\lceil \sqrt{\frac{n-2}{2}} \right\rceil . \tag{1}$$

By Step 4 and Lemmas 4 and 6, we have

$$length(H,\sigma) \leq \left\lceil \frac{n}{2} \right\rceil + 2\left\lceil \sqrt{\frac{n-2}{2}} \right\rceil . \tag{2}$$

Let $H^* = L \cup R \cup X^*$. That is, $H^* = H \cup \{(v_2, v_n)\}$, as illustrated by Fig. 3(a) and 3(b). By definition of $\sigma$ and Lemma 5, we have $\sigma(v_2) = 1$ and $\sigma(v_n) \geq \max_j \sigma(r_j)$. Therefore, any $\sigma$-increasing path of $H^*$ containing edge $(v_2, v_n)$ contains exactly one node of $R$, i.e., $v_n$, and thus has length at most $\lceil \frac{n}{2} \rceil$. It follows from Inequality (2) that

$$length(H^*,\sigma) \leq \left\lceil \frac{n}{2} \right\rceil + 2\left\lceil \sqrt{\frac{n-2}{2}} \right\rceil . \tag{3}$$

To prove Inequality (1), it remains to show that if $P$ is a $\sigma$-increasing path of $G$, then there is a $\sigma$-increasing path $Q$ of $H^*$ such that the length of $Q$ is no less than that of $P$. For each edge $(u, v)$ of $P$ with $\sigma(u) < \sigma(v)$, let $Q(u, v)$ be the $\sigma$-increasing path of $H^*$ defined as follows.

- If $u = \ell_i$ and $v = r_j$, then let $Q(u, v) = (u, v)$, which is a $\sigma$-increasing path of $X^*$.
- If $u = r_i$ and $v = \ell_j$, then let $Q(u, v) = (u, v)$, which is a $\sigma$-increasing path of $X^*$.
- If $u = \ell_i$ and $v = \ell_j$, then by $\sigma(\ell_i) < \sigma(\ell_j)$ we know $\psi_L(\ell_i) < \psi_L(\ell_j)$ and thus $i < j$. Let $Q(u, v) = (\ell_i, \ell_{i+1}, \ldots, \ell_j)$. Since $\sigma$ is a consistent ordering of $H$ with respect to $\mathcal{E}$, $Q(u, v)$ is a $\sigma$-increasing path of $L$.
- If $u = r_i$ and $v = r_j$, then by $\sigma(r_i) < \sigma(r_j)$ we know $\psi_R(r_i) < \psi_R(r_j)$ and thus $i < j$. Let $Q(u, v) = (r_i, r_{i+1}, \ldots, r_j)$. Since $\sigma$ is a consistent ordering of $H$ with respect to $\mathcal{E}$, $Q(u, v)$ is a $\sigma$-increasing path of $R$.

Let $Q$ be the union of $Q(u, v)$ for all edges $(u, v)$ of $P$. Since each $Q(u, v)$ is a $\sigma$-increasing path of $H^*$, so is $Q$. The length of $Q$ is no less than that of $P$. That is, we have

$$length(G,\sigma) \leq length(H^*,\sigma). \tag{4}$$

Since Inequality (1) is immediate from Inequalities (3) and (4), the lemma is proved. □

## 4 A Lower Bound

Let plane graph $N_k$ be defined recursively as follows.

- Let $N_1$ be the four-node internally triangulated plane graph with four external nodes.

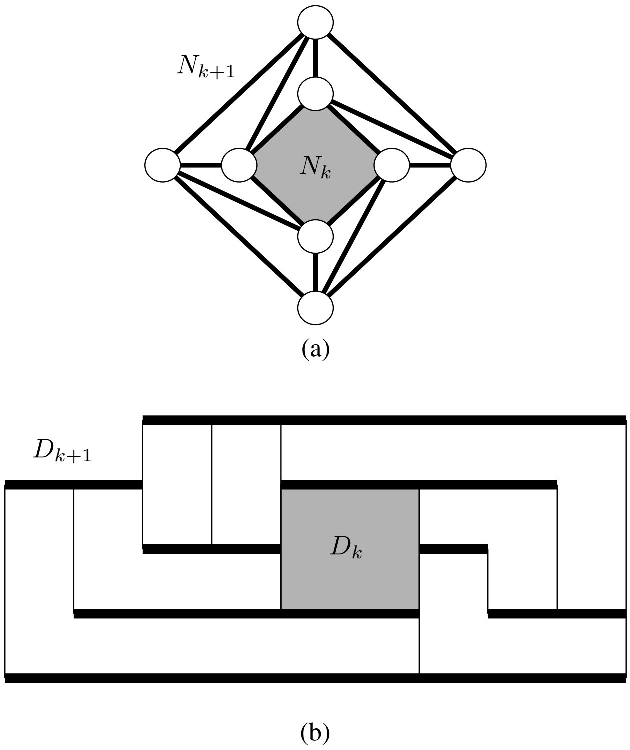



**Fig. 4.** (a) A four-connected plane graph $N_{k+1}$ and its relation with $N_k$. (b) A visibility representation $D_{k+1}$ of $N_{k+1}$ and its relation with $D_k$.

– Let $N_{k+1}$ be obtained from $N_k$ by adding four nodes and twelve edges in the way as shown in Fig. 4(a).

One can easily verify that each $N_k$ with $k \geq 1$ is indeed four-connected. The following lemma ensures that the the upper bound provided by Theorem 1 has an optimal first-order term.

**Lemma 8.** *All visibility representations of $N_k$ have heights at least $2k$.*

*Proof.* We prove the lemma by induction on $k$. The lemma holds trivially for $k = 1$. Assume for a contradiction that $N_{k+1}$ admits a visibility representation $D_{k+1}$ with height no more than $2k + 1$. Let $D_k$ be obtained from $D_{k+1}$ by deleting all the horizontal segments representing those four external nodes of $N_{k+1}$. Since $D_{k+1}$ has to reflect the planar embedding of $N_{k+1}$, $D_k$ is a visibility representation of $N_k$. Since the external nodes of $N_k$ are internal in $N_{k+1}$, the horizontal segments of $D_{k+1}$ representing the external nodes of $N_{k+1}$ have to wrap $D_k$ completely. That is, $D_{k+1}$ must have a horizontal segment above $D_k$ and a horizontal segment below $D_k$. Therefore, the height of $D_{k+1}$ is at least two more than that of $D_k$. It follows that the height of $D_k$ is at most $2k - 1$, contradicting the inductive hypothesis. Since $N_{k+1}$ cannot admit a visibility representation with height less than $2k + 2$, the lemma is proved. □

## 5 Concluding Remarks

It would be of interest to close the $\Theta(\sqrt{n})$ gap between the upper and lower bounds on the required height for the visibility representation of any $n$-node four-connected plane graph. We conjecture that the $\Theta(\sqrt{n})$ term in our upper bound can be reduced to $O(1)$.

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