Optimization and Machine Learning Solution

Midterm 1

November 10, 2010

- **Please give details of your calculation. A direct answer without explanation is not counted.**
- **Your answers must be in English.**
- **You can bring notes and the textbook. Other books or electronic devices are not allowed.**

Problem 1 (5%)

If C_1 and C_2 are convex sets in $\Re^n$ and we define

$$C_1 + C_2 \; = \; \{x_1 + x_2 |\; x_1 \in C_1,\; x_2 \in C_2\},$$

then is $C_1 + C_2$ a convex set?

Problem 2 (15%)

Consider $f(x) \; = \; e^x,\; x \in \Re$.

(a) What's the conjugate function $f^*(y)$ of $f(x)$?

(b) What's the conjugate function of $f^*(y)$?

Problem 3 (20%)

Consider the following problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1^2 - x_1 x_2 + 2x_2^2 - 4x_1 - 5x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 6 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) Is this a convex optimization problem?

(b) What's the KKT condition?

(c) Find the solution of this optimization problem.

Problem 4 (20%)

Consider the following problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 - 4 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) What's the optimal solution?

(b) Derive the dual problem.

(c) Solve the dual problem.

(d) Is there any gap between primal and dual optimal values?

Problem 5 (20%)

Let Q be an $n \times n$ symmetric and positive definite matrix and $\boldsymbol{p}$ be any $n \times 1$ vector. Consider the following two optimization problems:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \frac{1}{2}\boldsymbol{x}^T Q \boldsymbol{x} - \boldsymbol{p}^T \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{x} \succeq 0 \end{aligned} \tag{1}$$

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \frac{1}{2}\boldsymbol{x}^T Q \boldsymbol{x} \\ \text{subject to} \quad & Q\boldsymbol{x} \succeq \boldsymbol{p} \end{aligned} \tag{2}$$

($\succeq$ means component wise $\geq$)

Do these two problems have the same optimal solution or not?

Problem 6 (20%)

Consider a function of n variables,

$$f(\boldsymbol{p}) = -\sum_{i,j:i\neq j} n_{ij}(\log p_i - \log(p_i + p_j)),$$

where n_{ij} are non-negative constants and $p_i > 0$, $\forall i$.

(a) Is this function convex? If yes, give a proof. If not, give a counter example.

(b) If we replace p_i with e^{θ_i} (since $p_i > 0$) and obtain the following function

$$g(\boldsymbol{\theta}) \;=\; -\sum_{i,j:i\neq j} n_{ij}(\theta_i - \log(e^{\theta_i} + e^{\theta_j})),$$

is this function convex or not?