Delta Hedge

- The delta (hedge ratio) of a derivative f is defined as $\Delta \equiv \partial f / \partial S$.
- Thus $\Delta f \approx \Delta \times \Delta S$ for relatively small changes in the stock price, ΔS.
- A delta-neutral portfolio is hedged in the sense that it is immunized against small changes in the stock price.
- A trading strategy that dynamically maintains a delta-neutral portfolio is called delta hedge.

Delta Hedge (concluded)

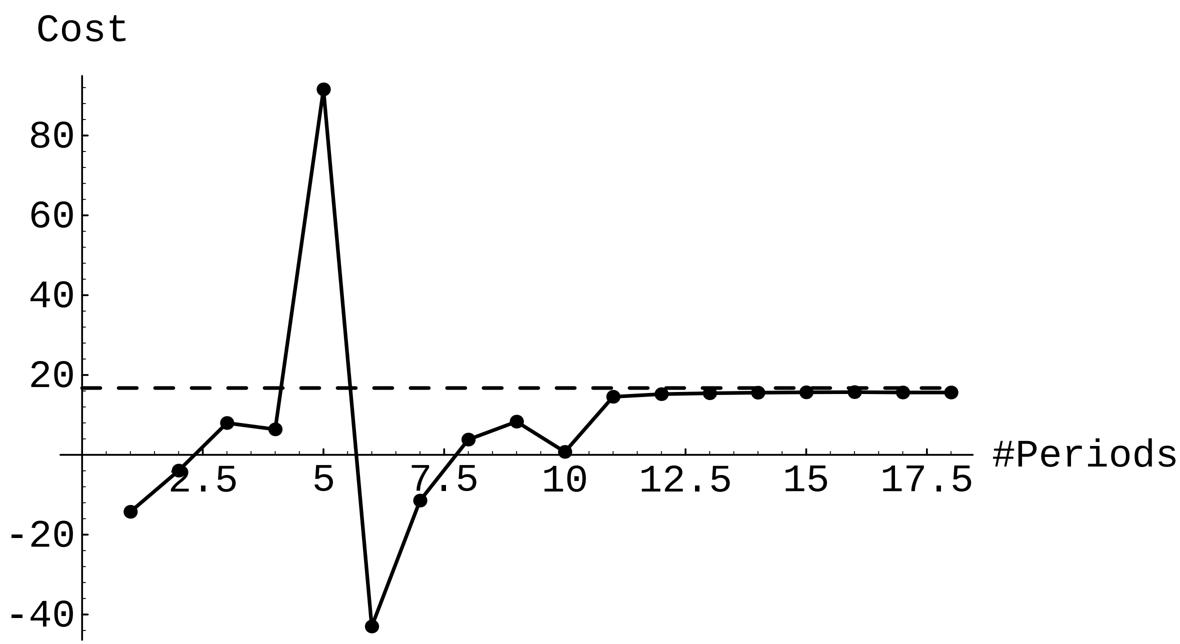
- Delta changes with the stock price.
- A delta hedge needs to be rebalanced periodically in order to maintain delta neutrality.
- In the limit where the portfolio is adjusted continuously, perfect hedge is achieved and the strategy becomes self-financing.
- This was the gist of the Black-Scholes-Merton argument.

Implementing Delta Hedge

- We want to hedge N short derivatives.
- Assume the stock pays no dividends.
- The delta-neutral portfolio maintains $N \times \Delta$ shares of stock plus B borrowed dollars such that

$$-N \times f + N \times \Delta \times S - B = 0.$$

- At next rebalancing point when the delta is Δ', buy $N \times (\Delta' - \Delta)$ shares to maintain $N \times \Delta'$ shares with a total borrowing of $B' = N \times \Delta' \times S' - N \times f'$.
- Delta hedge is the discrete-time analog of the continuous-time limit and will rarely be self-financing.


Cost
80
60
40
20
-20
-40
2.5
5
7.5
10
12.5
15
17.5
#Periods

Example

- A hedger is short 10,000 European calls.
- $\sigma = 30\%$ and $r = 6\%$.
- This call's expiration is four weeks away, its strike price is \$50, and each call has a current value of $f = 1.76791$.
- As an option covers 100 shares of stock, $N = 1{,}000{,}000$.
- The trader adjusts the portfolio weekly.
- The calls are replicated[a] well if the cumulative cost of trading *stock* is close to the call premium's FV.

---

[a] This example takes the replication viewpoint.

Example (continued)

- As $\Delta = 0.538560$, $N \times \Delta = 538,560$ shares are purchased for a total cost of $538{,}560 \times 50 = 26{,}928{,}000$ dollars to make the portfolio delta-neutral.
- The trader finances the purchase by borrowing

$$B = N \times \Delta \times S - N \times f = 25{,}160{,}090$$

dollars net.[a]

- The portfolio has zero net value now.

---

[a] This takes the hedging viewpoint — an alternative. See an exercise in the text.

Example (continued)

- At 3 weeks to expiration, the stock price rises to \$51.
- The new call value is $f' = 2.10580$.
- So the portfolio is worth

$$-N \times f' + 538{,}560 \times 51 - Be^{0.06/52} = 171,622$$

before rebalancing.

Example (continued)

- A delta hedge does not replicate the calls perfectly; it is not self-financing as \$171,622 can be withdrawn.
- The magnitude of the tracking error—the variation in the net portfolio value—can be mitigated if adjustments are made more frequently.
- In fact, the tracking error *over one rebalancing act* is positive about 68% of the time, but its expected value is essentially zero.[a]
- It is furthermore proportional to vega.

[a]Boyle and Emanuel (1980).

Example (continued)

- In practice tracking errors will cease to decrease beyond a certain rebalancing frequency.
- With a higher delta $\Delta' = 0.640355$, the trader buys $N \times (\Delta' - \Delta) = 101,795$ shares for \$5,191,545.
- The number of shares is increased to $N \times \Delta' = 640,355$.

Example (continued)

- The cumulative cost is

$$26{,}928{,}000 \times e^{0.06/52} + 5{,}191{,}545 = 32{,}150{,}634.$$

- The total borrowed amount is

$$B' = 640{,}355 \times 51 - N \times f' = 30{,}552{,}305.$$

- The portfolio is again delta-neutral with zero value.

		Option value	Delta	Change in delta	No. shares bought	Cost of shares	Cumulative cost
τ	S	f	Δ		$N \times (5)$	$(1) \times (6)$	$\text{FV}(8') + (7)$
	(1)	(2)	(3)	(5)	(6)	(7)	(8)
4	50	1.7679	0.53856	—	538,560	26,928,000	26,928,000
3	51	2.1058	0.64036	0.10180	101,795	5,191,545	32,150,634
2	53	3.3509	0.85578	0.21542	215,425	11,417,525	43,605,277
1	52	2.2427	0.83983	−0.01595	−15,955	−829,660	42,825,960
0	54	4.0000	1.00000	0.16017	160,175	8,649,450	51,524,853

The total number of shares is 1,000,000 at expiration (trading takes place at expiration, too).

Example (concluded)

- At expiration, the trader has 1,000,000 shares.
- They are exercised against by the in-the-money calls for \$50,000,000.
- The trader is left with an obligation of

$$51{,}524{,}853 - 50{,}000{,}000 = 1{,}524{,}853,$$

  which represents the replication cost.
- Compared with the FV of the call premium,

$$1{,}767{,}910 \times e^{0.06 \times 4/52} = 1{,}776{,}088,$$

  the net gain is $1{,}776{,}088 - 1{,}524{,}853 = 251{,}235$.

Tracking Error Revisited[a]

- The tracking error ϵ_n over n rebalancing acts (such as 251,235 above) has about the same probability of being positive as being negative.
- Subject to certain regularity conditions, the root-mean-square tracking error $\sqrt{E[\,\epsilon_n^2\,]}$ is $O(1/\sqrt{n}\,)$.[b]
- The root-mean-square tracking error increases with σ at first and then decreases.

---

[a] Bertsimas, Kogan, and Lo (2000).

[b] See also Grannan and Swindle (1996).

Delta-Gamma Hedge

- Delta hedge is based on the first-order approximation to changes in the derivative price, Δf, due to changes in the stock price, ΔS.
- When ΔS is not small, the second-order term, gamma $\Gamma \equiv \partial^2 f / \partial S^2$, helps (theoretically).
- A delta-gamma hedge is a delta hedge that maintains zero portfolio gamma, or gamma neutrality.
- To meet this extra condition, one more security needs to be brought in.

Delta-Gamma Hedge (concluded)

- Suppose we want to hedge short calls as before.
- A hedging call f_2 is brought in.
- To set up a delta-gamma hedge, we solve

$$\begin{aligned} -N \times f + n_1 \times S + n_2 \times f_2 - B &= 0 \quad \text{(self-financing)}, \\ -N \times \Delta + n_1 + n_2 \times \Delta_2 - 0 &= 0 \quad \text{(delta neutrality)}, \\ -N \times \Gamma + 0 + n_2 \times \Gamma_2 - 0 &= 0 \quad \text{(gamma neutrality)}, \end{aligned}$$

  for n_1, n_2, and B.
  - The gammas of the stock and bond are 0.

Other Hedges

- If volatility changes, delta-gamma hedge may not work well.
- An enhancement is the delta-gamma-vega hedge, which also maintains vega zero portfolio vega.
- To accomplish this, one more security has to be brought into the process.
- In practice, delta-vega hedge, which may not maintain gamma neutrality, performs better than delta hedge.

Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

And though the holes were rather small,
they had to count them all.
— The Beatles, *A Day in the Life* (1967)

The Combinatorial Method

- The combinatorial method can often cut the running time by an order of magnitude.
- The basic paradigm is to count the number of admissible paths that lead from the root to any terminal node.
- We first used this method in the linear-time algorithm for standard European option pricing on p. 231.
  - In general, it cannot apply to American options.
- We will now apply it to price barrier options.

The Reflection Principle[a]

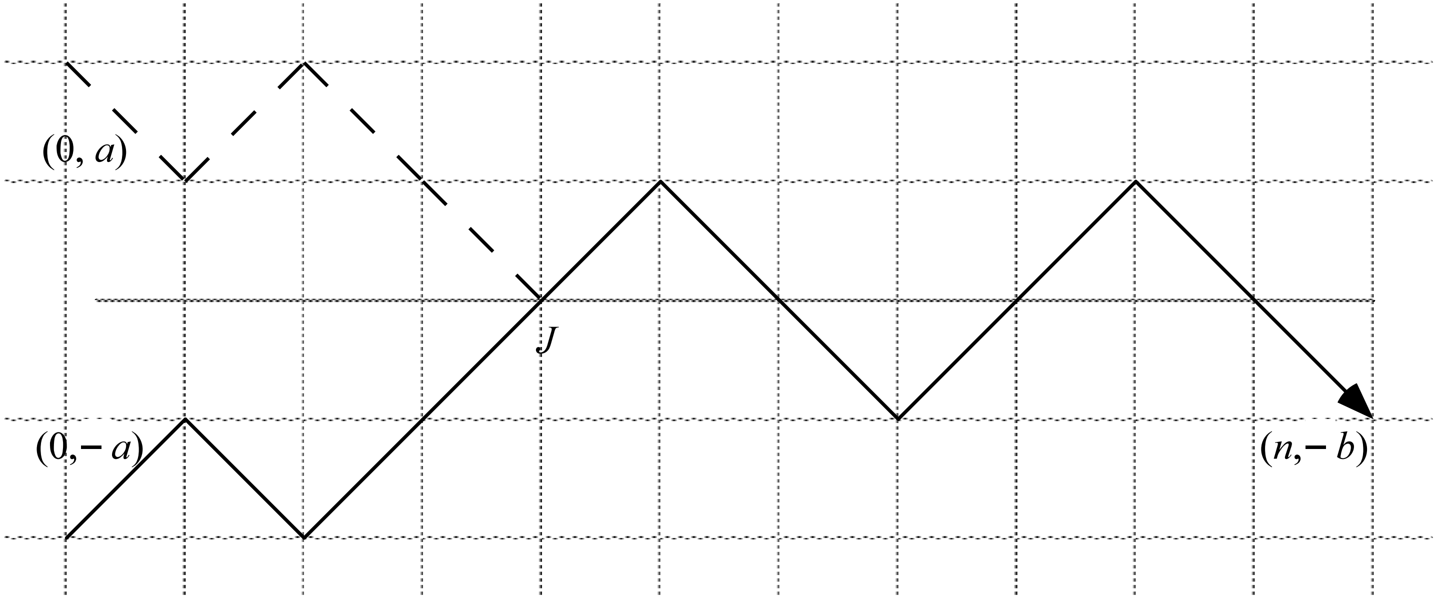
- Imagine a particle at position $(0, -\boldsymbol{a})$ on the integral lattice that is to reach $(n, -\boldsymbol{b})$.
- Without loss of generality, assume $\boldsymbol{a} > 0$ and $\boldsymbol{b} \geq 0$.
- This particle's movement:

$$
(i, j) \begin{array}{l} \nearrow (i+1, j+1) \quad \text{up move } S \to Su \\[1em] \searrow (i+1, j-1) \quad \text{down move } S \to Sd \end{array}
$$

- How many paths touch the x axis?

---

[a] André (1887).


(0, a)
J
(0,− a)
(n,− b)

The Reflection Principle (continued)

- For a path from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touches the x axis, let J denote the first point this happens.
- Reflect the portion of the path from $(0, -\boldsymbol{a})$ to J.
- A path from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$ is constructed.
- It also hits the x axis at J for the first time.
- The one-to-one mapping shows the number of paths from $(0, -\boldsymbol{a})$ to $(n, -\boldsymbol{b})$ that touch the x axis equals the number of paths from $(0, \boldsymbol{a})$ to $(n, -\boldsymbol{b})$.

The Reflection Principle (concluded)

- A path of this kind has $(n + \boldsymbol{b} + \boldsymbol{a})/2$ down moves and $(n - \boldsymbol{b} - \boldsymbol{a})/2$ up moves.
- Hence there are

$$\binom{n}{\frac{n+\boldsymbol{a}+\boldsymbol{b}}{2}} \tag{54}$$

  such paths for even $n + \boldsymbol{a} + \boldsymbol{b}$.
  - Convention: $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

Pricing Barrier Options (Lyuu, 1998)

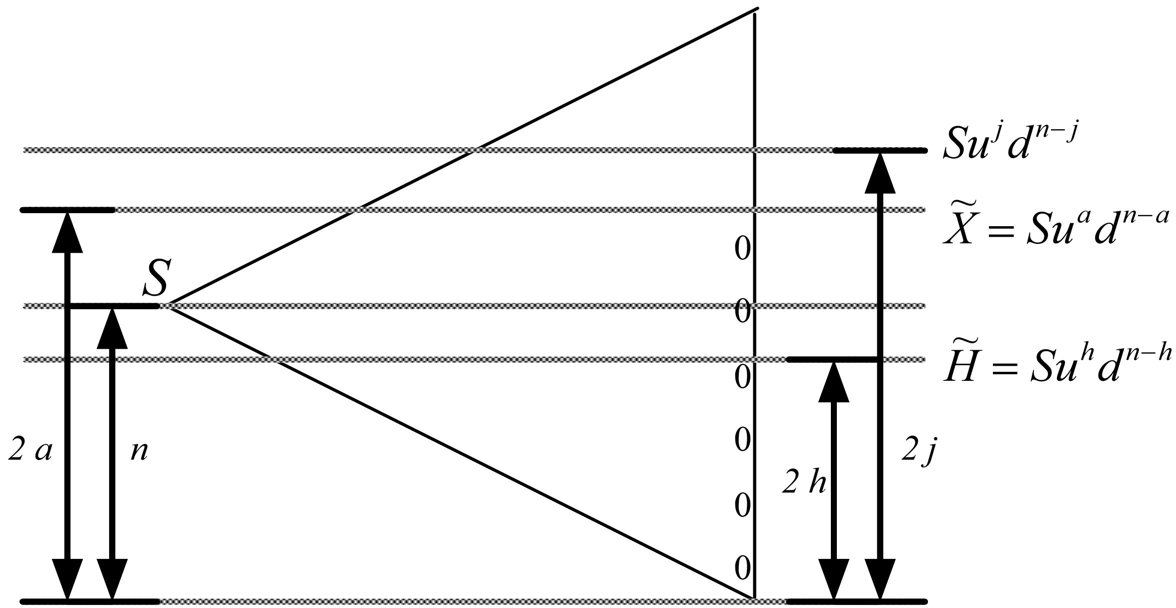
- Focus on the down-and-in call with barrier $H < X$.
- Assume $H < S$ without loss of generality.
- Define

$$\begin{aligned} a &\equiv \left\lceil \frac{\ln\left(X/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rceil = \left\lceil \frac{\ln(X/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rceil, \\ h &\equiv \left\lfloor \frac{\ln\left(H/\left(Sd^n\right)\right)}{\ln(u/d)} \right\rfloor = \left\lfloor \frac{\ln(H/S)}{2\sigma\sqrt{\Delta t}} + \frac{n}{2} \right\rfloor. \end{aligned}$$

  - h is such that $\tilde{H} \equiv S u^h d^{n-h}$ is the *terminal* price that is closest to, but does not exceed H.
  - a is such that $\tilde{X} \equiv S u^a d^{n-a}$ is the terminal price that is closest to, but is not exceeded by X.

Pricing Barrier Options (continued)

- The true barrier is replaced by the effective barrier $\tilde{H}$ in the binomial model.
- A process with n moves hence ends up in the money if and only if the number of up moves is at least a.
- The price $Su^k d^{n-k}$ is at a distance of $2k$ from the lowest possible price Sd^n on the binomial tree.
  -
  $$Su^k d^{n-k} = Sd^{-k} d^{n-k} = Sd^{n-2k}. \qquad (55)$$


$Su^j d^{n-j}$
$\widetilde{X} = Su^a d^{n-a}$
S
$\widetilde{H} = Su^h d^{n-h}$
0
0
0
0
0
0
2 a
n
2 h
2 j

Pricing Barrier Options (continued)

- The number of paths from S to the terminal price $Su^j d^{n-j}$ is $\binom{n}{j}$, each with probability $p^j(1-p)^{n-j}$.
- With reference to p. 525, the reflection principle can be applied with $\boldsymbol{a} = n - 2h$ and $\boldsymbol{b} = 2j - 2h$ in Eq. (54) on p. 522 by treating the S line as the x axis.
- Therefore,

$$\binom{n}{\frac{n+(n-2h)+(2j-2h)}{2}} = \binom{n}{n-2h+j}$$

paths hit $\tilde{H}$ in the process for $h \leq n/2$.

Pricing Barrier Options (concluded)

- The terminal price $Su^j d^{n-j}$ is reached by a path that hits the effective barrier with probability

$$\binom{n}{n-2h+j} p^j (1-p)^{n-j}.$$

- The option value equals

$$\frac{\sum_{j=a}^{2h} \binom{n}{n-2h+j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X\right)}{R^n}. \qquad (56)$$

  - $R \equiv e^{r\tau/n}$ is the riskless return per period.

- It implies a linear-time algorithm.

Convergence of BOPM

- Equation (56) results in the sawtooth-like convergence shown on p. 310.
- The reasons are not hard to see.
- The true barrier most likely does not equal the effective barrier.
- The same holds for the strike price and the effective strike price.
- The issue of the strike price is less critical.
- But the issue of the barrier is not negligible.

Convergence of BOPM (continued)

- Convergence is actually good if we limit n to certain values—191, for example.
- These values make the true barrier coincide with or occur just above one of the stock price levels, that is, $H \approx S d^j = S e^{-j\sigma\sqrt{\tau/n}}$ for some integer j.
- The preferred n's are thus

$$n = \left\lfloor \frac{\tau}{\left(\ln(S/H)/(j\sigma)\right)^2} \right\rfloor, \quad j = 1, 2, 3, \ldots$$

- There is only one minor technicality left.

Convergence of BOPM (continued)

- We picked the effective barrier to be one of the $n + 1$ possible terminal stock prices.
- However, the effective barrier above, Sd^j, corresponds to a terminal stock price only when $n - j$ is even by Eq. (55) on p. 524.[a]
- To close this gap, we decrement n by one, if necessary, to make $n - j$ an even number.

---

[a] We could have adopted the form Sd^j $(-n \leq j \leq n)$ for the effective barrier.

Convergence of BOPM (concluded)

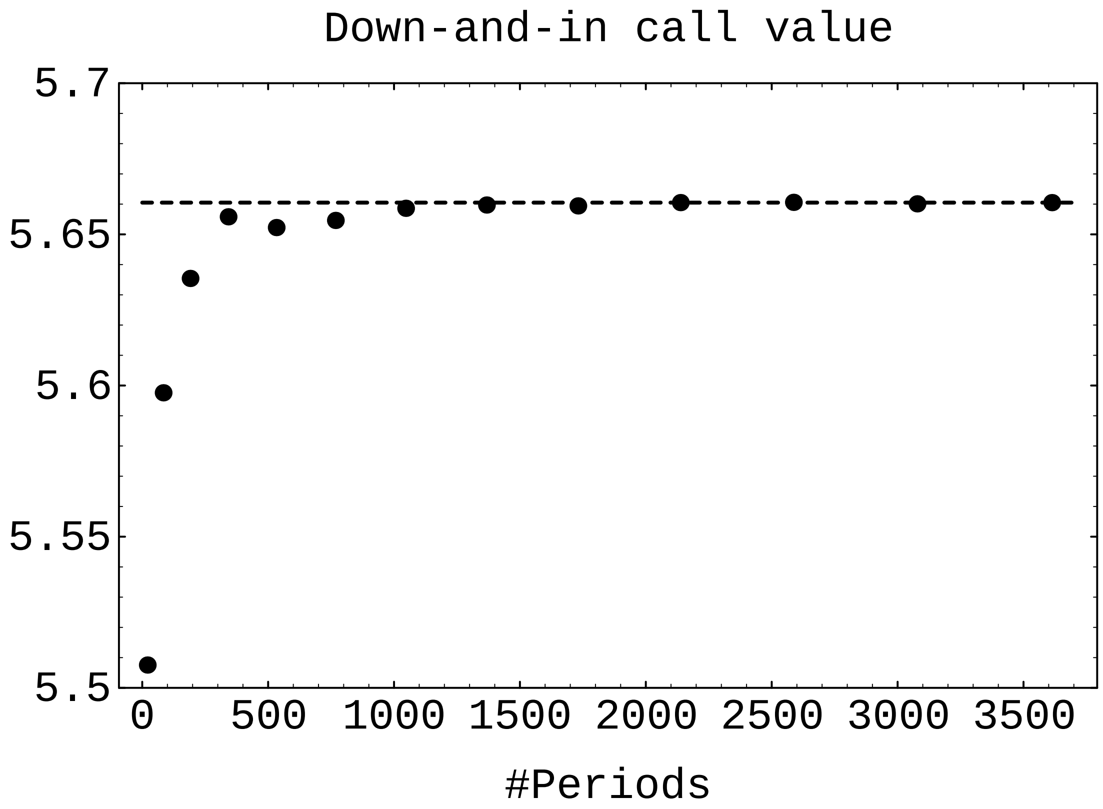
- The preferred n's are now

$$n = \begin{cases} \ell & \text{if } \ell - j \text{ is even} \\ \ell - 1 & \text{otherwise} \end{cases},$$

$j = 1, 2, 3, \ldots,$ where

$$\ell \equiv \left\lfloor \frac{\tau}{(\ln(S/H)/(j\sigma))^2} \right\rfloor.$$

- Evaluate pricing formula (56) on p. 527 only with the n's above.


Down-and-in call value
5.7
5.65
5.6
5.55
5.5
0 500 1000 1500 2000 2500 3000 3500
#Periods

Practical Implications

- Now that barrier options can be efficiently priced, we can afford to pick very large n's (p. 534).
- This has profound consequences.

n	Combinatorial method	
	Value	Time (milliseconds)
21	5.507548	0.30
84	5.597597	0.90
191	5.635415	2.00
342	5.655812	3.60
533	5.652253	5.60
768	5.654609	8.00
1047	5.658622	11.10
1368	5.659711	15.00
1731	5.659416	19.40
2138	5.660511	24.70
2587	5.660592	30.20
3078	5.660099	36.70
3613	5.660498	43.70
4190	5.660388	44.10
4809	5.659955	51.60
5472	5.660122	68.70
6177	5.659981	76.70
6926	5.660263	86.90
7717	5.660272	97.20

Practical Implications (concluded)

- Pricing is prohibitively time consuming when $S \approx H$ because $n \sim 1/\ln^2(S/H)$.
- This observation is indeed true of standard quadratic-time binomial tree algorithms.
- But it no longer applies to linear-time algorithms (p. 536).

	Barrier at 95.0			Barrier at 99.5			Barrier at 99.9	
n	Value	Time	n	Value	Time	n	Value	Time
	⋮		795	7.47761	8	19979	8.11304	253
2743	2.56095	31.1	3184	7.47626	38	79920	8.11297	1013
3040	2.56065	35.5	7163	7.47682	88	179819	8.11300	2200
3351	2.56098	40.1	12736	7.47661	166	319680	8.11299	4100
3678	2.56055	43.8	19899	7.47676	253	499499	8.11299	6300
4021	2.56152	48.1	28656	7.47667	368	719280	8.11299	8500
True	2.5615			7.4767			8.1130	

(All times in milliseconds.)

Trinomial Tree

- Set up a trinomial approximation to the geometric Brownian motion $dS/S = r\,dt + \sigma\,dW$.[a]
- The three stock prices at time Δt are S, Su, and Sd, where $ud = 1$.
- Impose the matching of mean and that of variance:

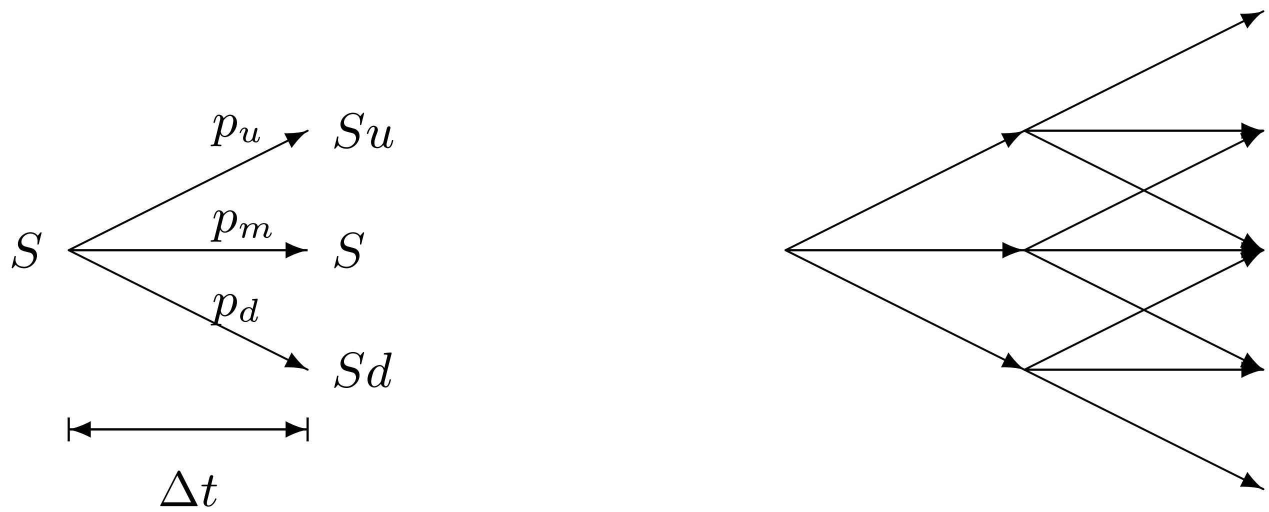
$$
\begin{aligned}
1 &= p_u + p_m + p_d, \\
SM &\equiv (p_u u + p_m + (p_d/u))\,S, \\
S^2 V &\equiv p_u(Su - SM)^2 + p_m(S - SM)^2 + p_d(Sd - SM)^2.
\end{aligned}
$$

[a]Boyle (1988).

- Above,

$$
\begin{aligned}
M &\equiv e^{r\Delta t},\\
V &\equiv M^2(e^{\sigma^2\Delta t}-1),
\end{aligned}
$$

by Eqs. (17) on p. 147.


p_u
Su
S
p_m
S
p_d
Sd
Δt

Trinomial Tree (continued)

- Use linear algebra to verify that

$$\begin{aligned} p_u &= \frac{u\left(V+M^2-M\right)-(M-1)}{(u-1)\left(u^2-1\right)}, \\ p_d &= \frac{u^2\left(V+M^2-M\right)-u^3(M-1)}{(u-1)\left(u^2-1\right)}. \end{aligned}$$

  – In practice, must make sure the probabilities lie between 0 and 1.

- Countless variations.

Trinomial Tree (concluded)

- Use $u = e^{\lambda\sigma\sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$\begin{aligned} p_u &\to \frac{1}{2\lambda^2} + \frac{\left(r + \sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}, \\ p_d &\to \frac{1}{2\lambda^2} - \frac{\left(r - 2\sigma^2\right)\sqrt{\Delta t}}{2\lambda\sigma}. \end{aligned}$$

- A nice choice for λ is $\sqrt{\pi/2}$.[a]

[a]Omberg (1988).

Barrier Options Revisited

- BOPM introduces a specification error by replacing the barrier with a nonidentical effective barrier.
- The trinomial model solves the problem by adjusting λ so that the barrier is hit exactly.[a]
- It takes

$$h = \frac{\ln(S/H)}{\lambda\sigma\sqrt{\Delta t}}$$

  consecutive down moves to go from S to H if h is an integer, which is easy to achieve by adjusting λ.
  - This is because $Se^{-h\lambda\sigma\sqrt{\Delta t}} = H$.

[a] Ritchken (1995).

Barrier Options Revisited (continued)

- Typically, we find the smallest $\lambda \geq 1$ such that h is an integer.
- That is, we find the largest integer $j \geq 1$ that satisfies $\frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}} \geq 1$ and then let

$$\lambda = \frac{\ln(S/H)}{j\sigma\sqrt{\Delta t}}.$$

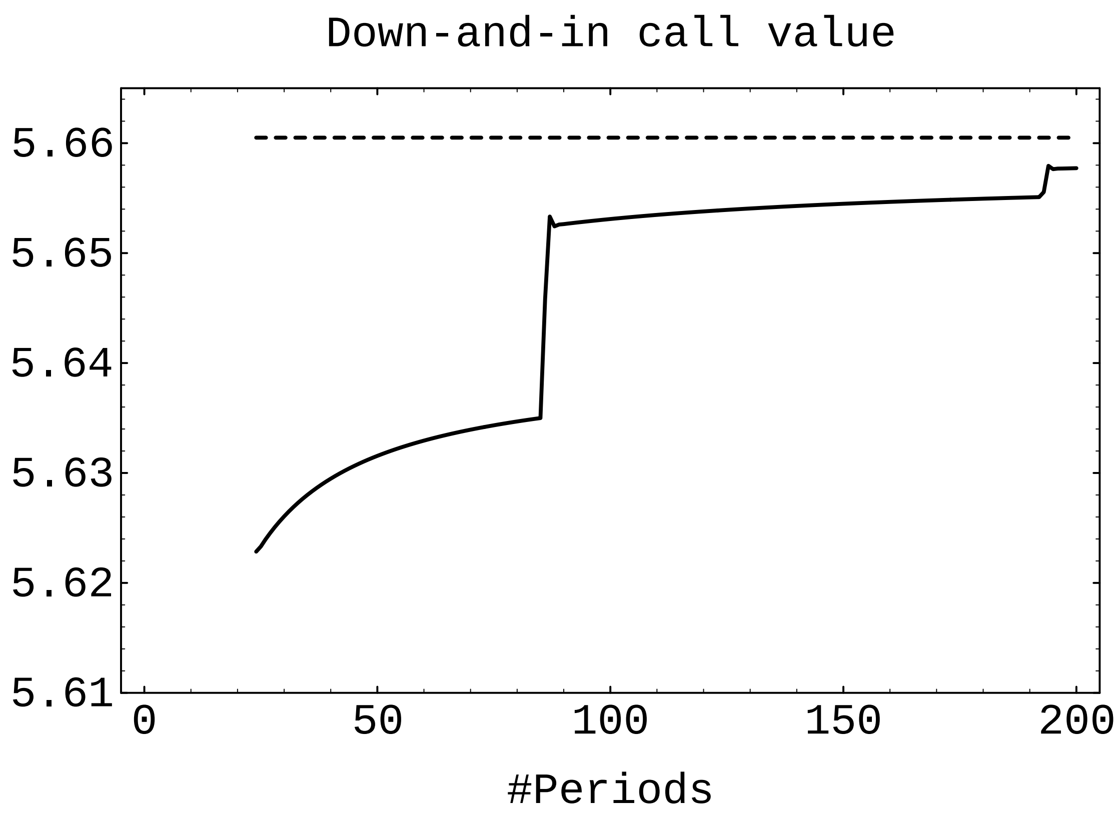
  - Such a λ may not exist for very small n's.
  - This is not hard to check.
- This done, one of the layers of the trinomial tree coincides with the barrier.

Barrier Options Revisited (concluded)

- The following probabilities may be used,

$$
\begin{aligned}
p_u &= \frac{1}{2\lambda^2} + \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}, \\
p_m &= 1 - \frac{1}{\lambda^2}, \\
p_d &= \frac{1}{2\lambda^2} - \frac{\mu' \sqrt{\Delta t}}{2\lambda\sigma}.
\end{aligned}
$$

  - $\mu' \equiv r - \sigma^2/2$.


Down-and-in call value
5.66
5.65
5.64
5.63
5.62
5.61
0
50
100
150
200
#Periods

Algorithms Comparison[a]

- So which algorithm is better, binomial or trinomial?
- Algorithms are often compared based on the n value at which they converge.
  - The one with the smallest n wins.
- So giraffes are faster than cheetahs because they take fewer strides to travel the same distance!
- Performance must be based on actual running times.

[a] Lyuu (1998).

Algorithms Comparison (concluded)

- Pages 310 and 545 show the trinomial model converges at a smaller n than BOPM.
- It is in this sense when people say trinomial models converge faster than binomial ones.
- But is the trinomial model better then?
- The linear-time binomial tree algorithm actually performs better than the trinomial one (see next page expanded from p. 534).

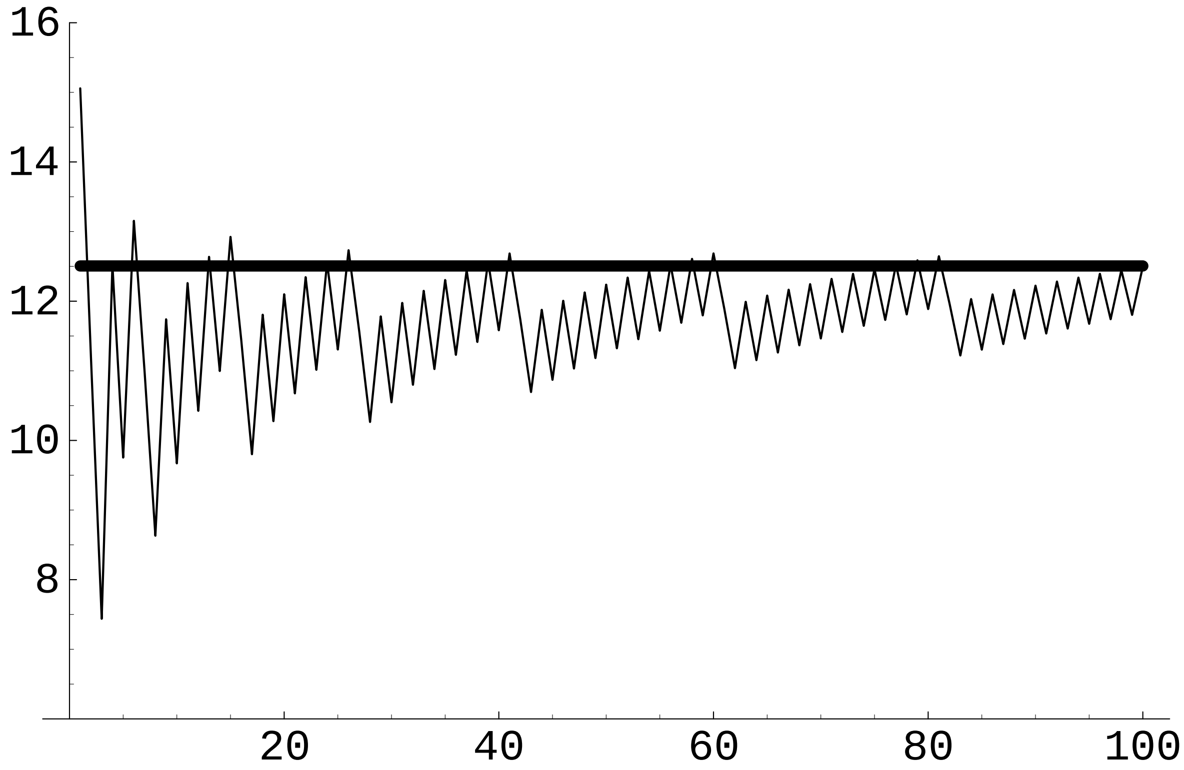
n	Combinatorial method		Trinomial tree algorithm	
	Value	Time	Value	Time
21	5.507548	0.30		
84	5.597597	0.90	5.634936	35.0
191	5.635415	2.00	5.655082	185.0
342	5.655812	3.60	5.658590	590.0
533	5.652253	5.60	5.659692	1440.0
768	5.654609	8.00	5.660137	3080.0
1047	5.658622	11.10	5.660338	5700.0
1368	5.659711	15.00	5.660432	9500.0
1731	5.659416	19.40	5.660474	15400.0
2138	5.660511	24.70	5.660491	23400.0
2587	5.660592	30.20	5.660493	34800.0
3078	5.660099	36.70	5.660488	48800.0
3613	5.660498	43.70	5.660478	67500.0
4190	5.660388	44.10	5.660466	92000.0
4809	5.659955	51.60	5.660454	130000.0
5472	5.660122	68.70		
6177	5.659981	76.70		

(All times in milliseconds.)

Double-Barrier Options

- Double-barrier options are barrier options with two barriers $L < H$.
- Assume $L < S < H$.
- The binomial model produces oscillating option values (see plot next page).[a]

[a] Chao (1999); Dai and Lyuu (2005);


16
14
12
10
8
20
40
60
80
100

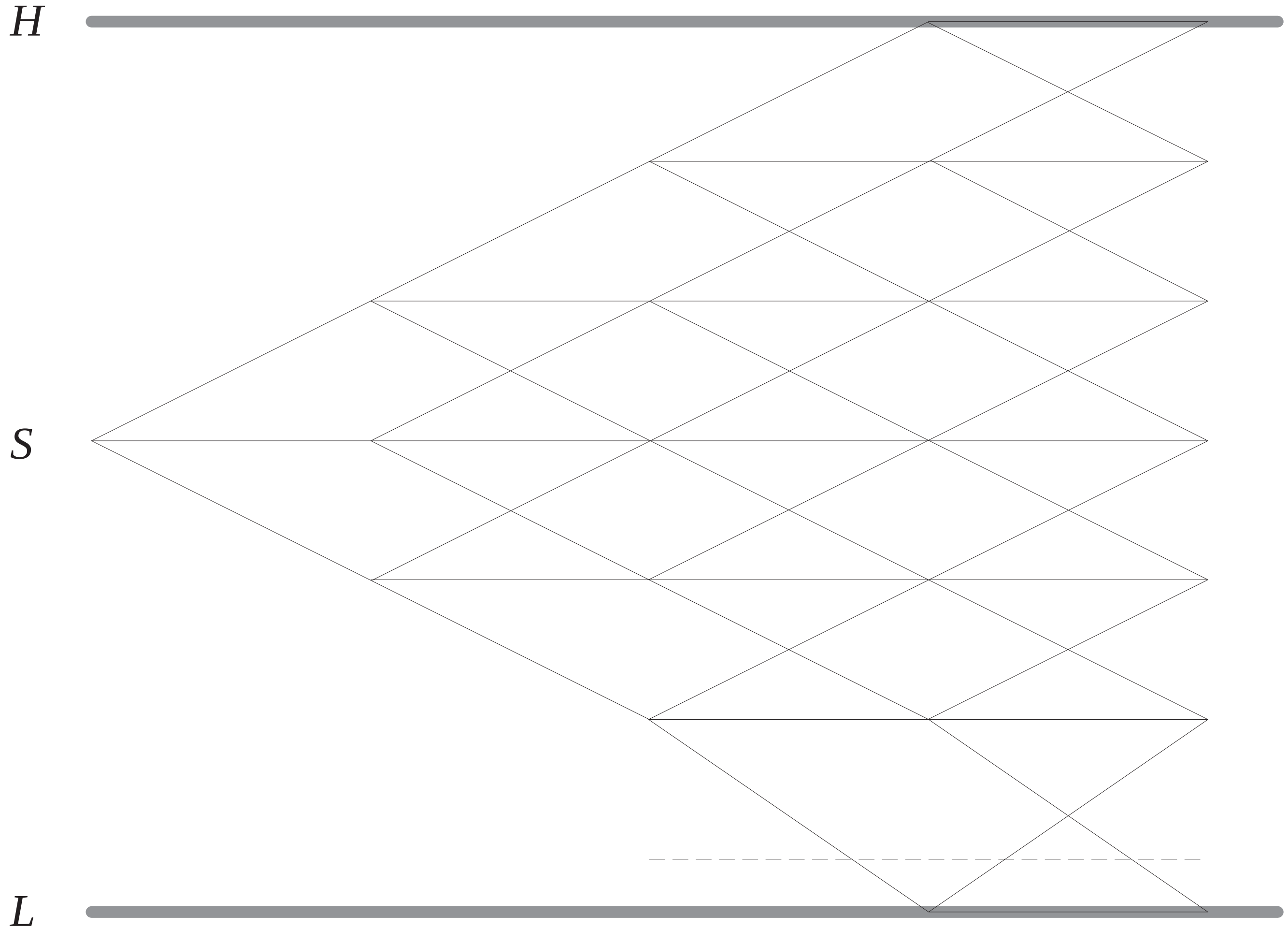
Double-Barrier Knock-Out Options

- We knew how to pick the λ so that one of the layers of the trinomial tree coincides with one barrier, say H.
- This choice, however, does not guarantee that the other barrier, L, is also hit.
- One way to handle this problem is to lower the layer of the tree just above L to coincide with L.[a]
  - More general ways to make the trinomial model hit both barriers are available.[b]

---

[a] Ritchken (1995).

[b] Hsu and Lyuu (2006). Dai and Lyuu (2006) combine binomial and trinomial trees to derive an $O(n)$-time algorithm for double-barrier options!


H
S
L

Double-Barrier Knock-Out Options (continued)

- The probabilities of the nodes on the layer above L must be adjusted.
- Let ℓ be the positive integer such that

$$Sd^{\ell+1} < L < Sd^{\ell}.$$

- Hence the layer of the tree just above L has price Sd^{ℓ}.

Double-Barrier Knock-Out Options (concluded)

- Define $\gamma > 1$ as the number satisfying

$$L = Sd^{\ell-1} e^{-\gamma\lambda\sigma\sqrt{\Delta t}}.$$

  - The prices between the barriers are

$$L, Sd^{\ell-1}, \ldots, Sd^2, Sd, S, Su, Su^2, \ldots, Su^{h-1}, Su^h = H.$$

- The probabilities for the nodes with price equal to $Sd^{\ell-1}$ are

$$p'_u = \frac{b + a\gamma}{1+\gamma}, \quad p'_d = \frac{b-a}{\gamma+\gamma^2}, \quad \text{and} \quad p'_m = 1 - p'_u - p'_d,$$

where $a \equiv \mu'\sqrt{\Delta t}/(\lambda\sigma)$ and $b \equiv 1/\lambda^2$.

Convergence: Binomial vs. Trinomial