The Jacobi Symbol[a]

- The Legendre symbol only works for odd *prime* moduli.
- The **Jacobi symbol** $(a \mid m)$ extends it to cases where m is not prime.
  - a is sometimes called the **numerator** and m the **denominator**.
- Trivially, $(1 \mid m) = 1$.
- Define $(a \mid 1) = 1$.

[a] Carl Jacobi (1804–1851).

The Jacobi Symbol (concluded)

- Let $m = p_1 p_2 \cdots p_k$ be the prime factorization of m.
- When $m > 1$ is odd and $\gcd(a, m) = 1$, then

$$(a \,|\, m) \stackrel{\Delta}{=} \prod_{i=1}^{k} (a \,|\, p_i).$$

  - Note that the Jacobi symbol equals ± 1.
  - It reduces to the Legendre symbol when m is a prime.

Properties of the Jacobi Symbol

The Jacobi symbol has the following properties when it is defined.

1. $(ab \,|\, m) = (a \,|\, m)(b \,|\, m)$.
2. $(a \,|\, m_1 m_2) = (a \,|\, m_1)(a \,|\, m_2)$.
3. If $a \equiv b \bmod m$, then $(a \,|\, m) = (b \,|\, m)$.
4. $(-1 \,|\, m) = (-1)^{(m-1)/2}$ (by Lemma 70 on p. 581).
5. $(2 \,|\, m) = (-1)^{(m^2-1)/8}$.[a]
6. If a and m are both odd, then $(a \,|\, m)(m \,|\, a) = (-1)^{(a-1)(m-1)/4}$.

---

[a] By Lemma 70 (p. 581) and some parity arguments.

Properties of the Jacobi Symbol (concluded)

- Properties 3–6 allow us to calculate the Jacobi symbol *without* factorization.
  - It will also yield the same result as Euler's test[a] when m is an odd prime.
- This situation is similar to the Euclidean algorithm.
- Note also that $(a \mid m) = 1/(a \mid m)$ because $(a \mid m) = \pm 1$.[b]

---

[a] Recall p. 573.

[b] Contributed by Mr. Huang, Kuan-Lin (`B96902079`, `R00922018`) on December 6, 2011.

Calculation of $(2200 \mid 999)$

$$
\begin{aligned}
(2200 \mid 999) &= (202 \mid 999) \\
&= (2 \mid 999)(101 \mid 999) \\
&= (-1)^{(999^2-1)/8}(101 \mid 999) \\
&= (-1)^{124750}(101 \mid 999) = (101 \mid 999) \\
&= (-1)^{(100)(998)/4}(999 \mid 101) = (-1)^{24950}(999 \mid 101) \\
&= (999 \mid 101) = (90 \mid 101) = (-1)^{(101^2-1)/8}(45 \mid 101) \\
&= (-1)^{1275}(45 \mid 101) = -(45 \mid 101) \\
&= -(-1)^{(44)(100)/4}(101 \mid 45) = -(101 \mid 45) = -(11 \mid 45) \\
&= -(-1)^{(10)(44)/4}(45 \mid 11) = -(45 \mid 11) \\
&= -(1 \mid 11) = -1.
\end{aligned}
$$

A Result Generalizing Proposition 10.3 in the Textbook

Theorem 72 *The group of set* $\Phi(n)$ *under multiplication* $\bmod n$ *has a primitive root if and only if* n *is either 1, 2, 4,* p^k*, or* $2p^k$ *for some nonnegative integer* k *and an odd prime* p*.*

This result is essential in the proof of the next lemma.

The Jacobi Symbol and Primality Test[a]

Lemma 73 *If* $(M \,|\, N) \equiv M^{(N-1)/2} \bmod N$ *for all* $M \in \Phi(N)$*, then* N *is a prime. (Assume* N *is odd.)*

- Assume $N = mp$, where p is an odd prime, $\gcd(m, p) = 1$, and $m > 1$ (not necessarily prime).
- Let $r \in \Phi(p)$ such that $(r \,|\, p) = -1$.
- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that

$$
\begin{aligned}
M &= r \bmod p, \\
M &= 1 \bmod m.
\end{aligned}
$$

[a] Mr. Clement Hsiao (`B4506061`, `R88526067`) pointed out that the textbook's proof for Lemma 11.8 is incorrect in January 1999 while he was a senior.

The Proof (continued)

- By the hypothesis,

$$M^{(N-1)/2} = (M \mid N) = (M \mid p)(M \mid m) = -1 \bmod N.$$

- Hence

$$M^{(N-1)/2} = -1 \bmod m.$$

- But because $M = 1 \bmod m$,

$$M^{(N-1)/2} = 1 \bmod m,$$

  a contradiction.

The Proof (continued)

- Second, assume that $N = p^a$, where p is an odd prime and $a \geq 2$.
- By Theorem 72 (p. 596), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- As $r \in \Phi(N)$ (prove it), we have

$$r^{N-1} = 1 \bmod N.$$

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \,|\, (N-1),$$

  which implies that $p \,|\, (N-1)$.[a]

- But this is impossible given that $p \,|\, N$.

[a] For $p-1$ divides $N-1 = p^a - 1$.

The Proof (continued)

- Third, assume that $N = mp^a$, where p is an odd prime, $\gcd(m, p) = 1$, $m > 1$ (not necessarily prime), and a is even.
- The proof mimics that of the second case.
- By Theorem 72 (p. 596), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- In particular,
$$M^{N-1} = 1 \bmod p^a \tag{15}$$
for all $M \in \Phi(N)$.

- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that
$$\begin{aligned} M &= r \bmod p^a, \\ M &= 1 \bmod m. \end{aligned}$$

- Because $M = r \bmod p^a$ and Eq. (15),
$$r^{N-1} = 1 \bmod p^a.$$

The Proof (concluded)

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \mid (N-1),$$

  which implies that $p \mid (N-1)$.

- But this is impossible given that $p \mid N$.

The Number of Witnesses to Compositeness

Theorem 74 (Solovay & Strassen, 1977) *If N is an odd composite, then $(M \mid N) \equiv M^{(N-1)/2} \bmod N$ for at most half of $M \in \Phi(N)$.*

- By Lemma 73 (p. 597) there is at least one $a \in \Phi(N)$ such that $(a \mid N) \not\equiv a^{(N-1)/2} \bmod N$.
- Let $B \stackrel{\Delta}{=} \{ b_1, b_2, \ldots, b_k \} \subseteq \Phi(N)$ be the set of *all* distinct residues such that $(b_i \mid N) \equiv b_i^{(N-1)/2} \bmod N$.
- Let $aB \stackrel{\Delta}{=} \{ ab_i \bmod N : i = 1, 2, \ldots, k \}$.
- Clearly, $aB \subseteq \Phi(N)$, too.

The Proof (concluded)

- $|\, aB \,| = k$.
  - $ab_i \equiv ab_j \bmod N$ implies $N \mid a(b_i - b_j)$, which is impossible because $\gcd(a, N) = 1$ and $N > |\, b_i - b_j \,|$.
- $aB \cap B = \emptyset$ because

$$\begin{aligned} (ab_i)^{(N-1)/2} \bmod 2 &= a^{(N-1)/2} b_i^{(N-1)/2} \bmod 2 \\ &\neq (a \mid N)(b_i \mid N) = (ab_i \mid N). \end{aligned}$$

- Combining the above two results, we know

$$\frac{|\, B \,|}{\phi(N)} \leq \frac{|\, B \,|}{|\, B \cup aB \,|} = 0.5.$$

```
if N is even but N ≠ 2 then
    return "N is composite";
else if N = 2 then
    return "N is a prime";
end if
Pick M ∈ { 2, 3, ..., N − 1 } randomly;
if gcd(M, N) > 1 then
    return "N is composite";
else
    if (M | N) ≡ M^((N−1)/2) mod N then
        return "N is (probably) a prime";
    else
        return "N is composite";
    end if
end if
```

Analysis

- The algorithm certainly runs in polynomial time.
- There are no false positives (for COMPOSITENESS).
  - When the algorithm says the number is composite, it is always correct.

Analysis (concluded)

- The probability of a false negative (again, for COMPOSITENESS) is at most one half.
  - Suppose the input is composite.
  - By Theorem 74 (p. 604),

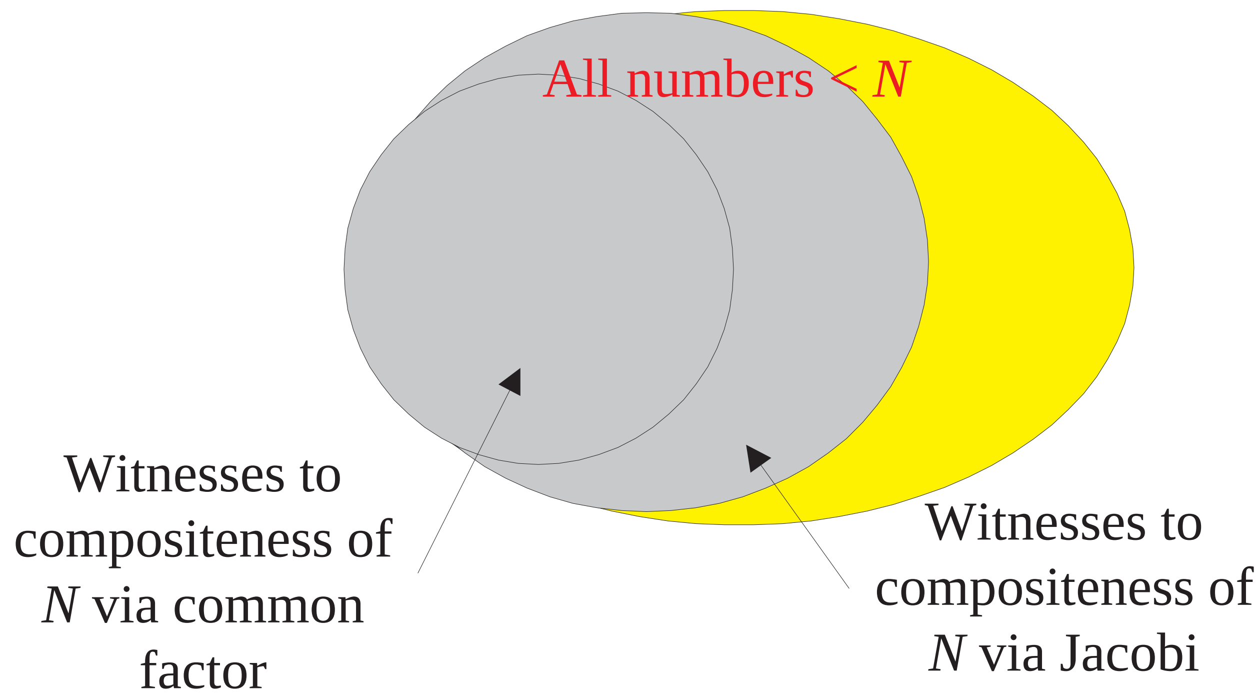
$$\text{prob}[\,\text{algorithm answers “no”} \mid N \text{ is composite}\,] \leq 0.5.$$

  - Note that we are not referring to the probability that N is composite when the algorithm says “no.”
- So it is a Monte Carlo algorithm for COMPOSITENESS[a] by the definition on p. 551.

---

[a]Not PRIMES.

The Improved Density Attack for COMPOSITENESS



Randomized Complexity Classes; RP

- Let N be a polynomial-time precise NTM that runs in time $p(n)$ and has 2 nondeterministic choices at each step.
- N is a **polynomial Monte Carlo Turing machine** for a language L if the following conditions hold:
  - If $x \in L$, then at least half of the $2^{p(n)}$ computation paths of N on x halt with "yes" where $n = |\, x \,|$.
  - If $x \notin L$, then all computation paths halt with "no."
- The class of all languages with polynomial Monte Carlo TMs is denoted **RP** (**randomized polynomial time**).[a]

[a] Adleman & Manders (1977).

Comments on RP

- In analogy to Proposition 41 (p. 344), a "yes" instance of an RP problem has many certificates (witnesses).
- There are no false positives.
- If we associate nondeterministic steps with flipping fair coins, then we can phrase RP in the language of probability.
  - If $x \in L$, then $N(x)$ halts with "yes" with probability at least 0.5.
  - If $x \notin L$, then $N(x)$ halts with "no."

Comments on RP (concluded)

- The probability of false negatives is ≤ 0.5.
- But *any* constant ϵ between 0 and 1 can replace 0.5.
  - Repeat the algorithm

$$k \triangleq \left\lceil -\frac{1}{\log_2 \epsilon} \right\rceil$$

    times.
  - Answer "no" only if all the runs answer "no."
  - The probability of false negatives becomes $\epsilon^k \leq 0.5$.

Where RP Fits

- $P \subseteq RP \subseteq NP$.
  - A deterministic TM is like a Monte Carlo TM except that all the coin flips are ignored.
  - A Monte Carlo TM is an NTM with more demands on the number of accepting paths.
- COMPOSITENESS $\in$ RP;[a] PRIMES $\in$ coRP; PRIMES $\in$ RP.[b]
  - In fact, PRIMES $\in$ P.[c]
- $RP \cup coRP$ is an alternative "plausible" notion of efficient computation.

---

[a] Rabin (1976); Solovay & Strassen (1977).
[b] Adleman & Huang (1987).
[c] Agrawal, Kayal, & Saxena (2002).

ZPP[a] (Zero Probabilistic Polynomial)

- The class **ZPP** is defined as RP $\cap$ coRP.
- A language in ZPP has *two* Monte Carlo algorithms, one with no false positives (RP) and the other with no false negatives (coRP).
- If we repeatedly run both Monte Carlo algorithms, *eventually* one definite answer will come (unlike RP).
  - A *positive* answer from the one without false positives.
  - A *negative* answer from the one without false negatives.

[a]Gill (1977).

The ZPP Algorithm (**Las Vegas**)

{Suppose $L \in$ ZPP.}
{N_1 has no false positives, and N_2 has no false negatives.}
while `true` **do**
    if $N_1(x) =$ "yes" **then**
        return "yes";
    end if
    if $N_2(x) =$ "no" **then**
        return "no";
    end if
end while

ZPP (concluded)

- The *expected* running time for the correct answer to emerge is polynomial.
  - The probability that a run of the 2 algorithms does not generate a definite answer is 0.5 (why?).
  - Let $p(n)$ be the running time of each run of the while-loop.
  - The expected running time for a definite answer is

$$\sum_{i=1}^{\infty} 0.5^i i p(n) = 2p(n).$$

- Essentially, ZPP is the class of problems that can be solved, without errors, in expected polynomial time.

Large Deviations

- Suppose you have a *biased* coin.
- One side has probability $0.5 + \epsilon$ to appear and the other $0.5 - \epsilon$, for some $0 < \epsilon < 0.5$.
- But you do not know which is which.
- How to decide which side is the more likely side—with high confidence?
- Answer: Flip the coin many times and pick the side that appeared the most times.
- Question: Can you quantify your confidence?

The (Improved) Chernoff Bound[a]

Theorem 75 (Chernoff, 1952) *Suppose* $x_1, x_2, \ldots, x_n$ *are independent random variables taking the values 1 and 0 with probabilities* p *and* $1-p$*, respectively. Let* $X = \sum_{i=1}^n x_i$*. Then for any constant* $0 \le \theta \le 1$*,*

$$\text{prob}[\, X \geq (1+\theta)\, pn \,] \leq e^{-\theta^2 pn/3}.$$

- The probability that the deviate of a **binomial random variable** from its expected value $E[\, X\,] = E\,[\, \sum_{i=1}^n x_i \,] = pn$ decreases exponentially with the deviation.

[a]Herman Chernoff (1923–). This bound is asymptotically optimal. The original bound is $e^{-2\theta^2 p^2 n}$ (McDiarmid, 1998).

The Proof

- Let t be any positive real number.
- Then
$$\text{prob}[\, X \geq (1+\theta)\, pn\,] = \text{prob}[\, e^{tX} \geq e^{t(1+\theta)\, pn}\,].$$
- Markov's inequality (p. 554) generalized to real-valued random variables says that
$$\text{prob}\left[\, e^{tX} \geq kE[\, e^{tX}\,]\right] \leq 1/k.$$
- With $k = e^{t(1+\theta)\, pn} / E[\, e^{tX}\,]$, we have[a]
$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{-t(1+\theta)\, pn} E[\, e^{tX}\,].$$

---

[a]Note that X does not appear in k. Contributed by Mr. Ao Sun (`R05922147`) on December 20, 2016.

The Proof (continued)

- Because $X = \sum_{i=1}^{n} x_i$ and x_i's are independent,

$$E[\, e^{tX} \,] = (E[\, e^{tx_1} \,])^n = [\, 1 + p(e^t - 1) \,]^n.$$

- Substituting, we obtain

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn \,] &\leq e^{-t(1+\theta)\, pn} [\, 1 + p(e^t - 1) \,]^n \\ &\leq e^{-t(1+\theta)\, pn} e^{pn(e^t - 1)} \end{aligned}$$

as $(1+a)^n \leq e^{an}$ for all $a > 0$.

The Proof (concluded)

- With the choice of $t = \ln(1+\theta)$, the above becomes

$$\text{prob}[\, X \geq (1+\theta)\, pn \,] \leq e^{pn[\,\theta-(1+\theta)\ln(1+\theta)\,]}.$$

- The exponent expands to[a]

$$-\frac{\theta^2}{2} + \frac{\theta^3}{6} - \frac{\theta^4}{12} + \cdots$$

for $0 \leq \theta \leq 1$.

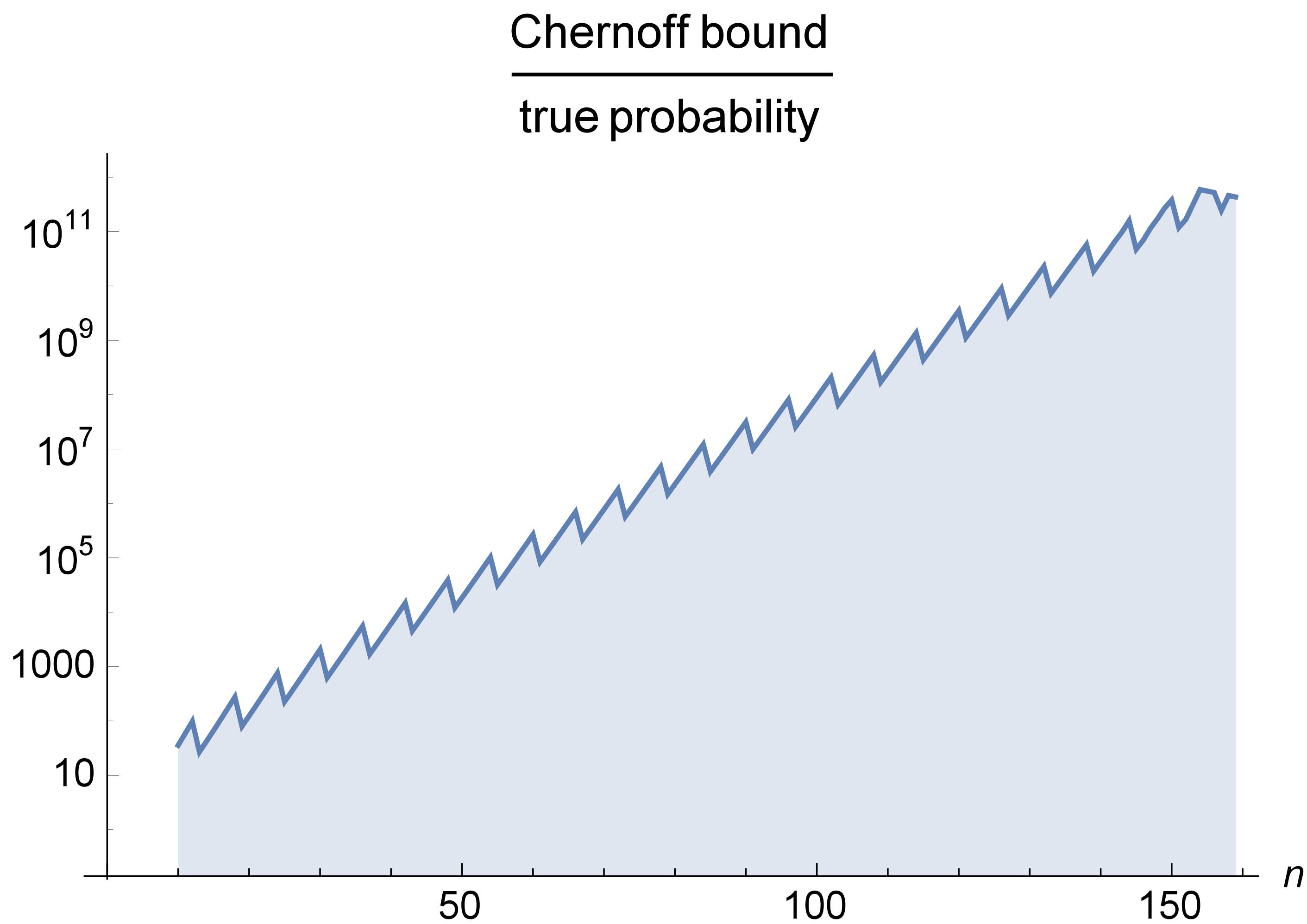
- But it is less than

$$-\frac{\theta^2}{2} + \frac{\theta^3}{6} \leq \theta^2 \left(-\frac{1}{2} + \frac{\theta}{6} \right) \leq \theta^2 \left(-\frac{1}{2} + \frac{1}{6} \right) = -\frac{\theta^2}{3}.$$

---

[a] Or McDiarmid (1998): $x - (1+x)\ln(1+x) \leq -3x^2/(6+2x)$ for all $x \geq 0$.

How Good Is the Bound?



Other Variations of the Chernoff Bound

The following can be proved similarly (prove it).

Theorem 76 *Given the same terms as Theorem 75 (p. 618),*

$$\text{prob}[\, X \leq (1-\theta)\, pn \,] \leq e^{-\theta^2 pn/2}.$$

The following slightly looser inequalities achieve symmetry.

Theorem 77 (Karp, Luby, & Madras, 1989) *Given the same terms as Theorem 75 (p. 618) except with* $0 \leq \theta \leq 2$,

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn \,] &\leq e^{-\theta^2 pn/4}, \\ \text{prob}[\, X \leq (1-\theta)\, pn \,] &\leq e^{-\theta^2 pn/4}. \end{aligned}$$

Power of the Majority Rule

The next result follows from Theorem 76 (p. 623).

Corollary 78 *If $p = (1/2) + \epsilon$ for some $0 \leq \epsilon \leq 1/2$, then*

$$\text{prob}\left[\sum_{i=1}^{n} x_i \leq n/2\right] \leq e^{-\epsilon^2 n/2}.$$

- The textbook's corollary to Lemma 11.9 seems too loose, at $e^{-\epsilon^2 n/6}$.[a]
- Our original problem (p. 617) hence demands, e.g., $n \approx 1.4k/\epsilon^2$ independent coin flips to guarantee making an error with probability $\leq 2^{-k}$ with the majority rule.

---

[a]See Dubhashi & Panconesi (2012) for many Chernoff-type bounds.

BPP[a] (Bounded Probabilistic Polynomial)

- The class **BPP** contains all languages L for which there is a precise polynomial-time NTM N such that:
  - If $x \in L$, then at least $3/4$ of the computation paths of N on x lead to "yes."
  - If $x \notin L$, then at least $3/4$ of the computation paths of N on x lead to "no."
- So N accepts or rejects by a *clear* majority.

[a] Gill (1977).

Magic $3/4$?

- The number $3/4$ bounds the probability (ratio) of a right answer away from $1/2$.
- Any constant *strictly* between $1/2$ and 1 can be used without affecting the class BPP.
- In fact, as with RP,

$$\frac{1}{2} + \frac{1}{q(n)}$$

for any polynomial $q(n)$ can replace $3/4$.

- The next algorithm shows why.

The Majority Vote Algorithm

Suppose L is decided by N by majority $(1/2) + \epsilon$.

for $i = 1, 2, \ldots, 2k + 1$ **do**
    Run N on input x;
end for
if "yes" is the majority answer **then**
    "yes";
else
    "no";
end if

Analysis

- By Corollary 78 (p. 624), the probability of a false answer is at most $e^{-\epsilon^2 k}$.
- By taking $k = \lceil 2/\epsilon^2 \rceil$, the error probability is at most $e^{-2} < 1/4$.
- Even if ϵ is any inverse polynomial, k remains a polynomial in n.
- The running time remains polynomial: $2k+1$ times N's running time.

Aspects of BPP

- BPP is the most comprehensive yet plausible notion of efficient computation.
  - If a problem is in BPP, we take it to mean that the problem can be solved efficiently.
  - In this aspect, BPP has effectively replaced P.
- $(\text{RP} \cup \text{coRP}) \subseteq (\text{NP} \cup \text{coNP})$.
- $(\text{RP} \cup \text{coRP}) \subseteq \text{BPP}$.
- Whether $\text{BPP} \subseteq (\text{NP} \cup \text{coNP})$ is unknown.
- But it is unlikely that $\text{NP} \subseteq \text{BPP}$.[a]

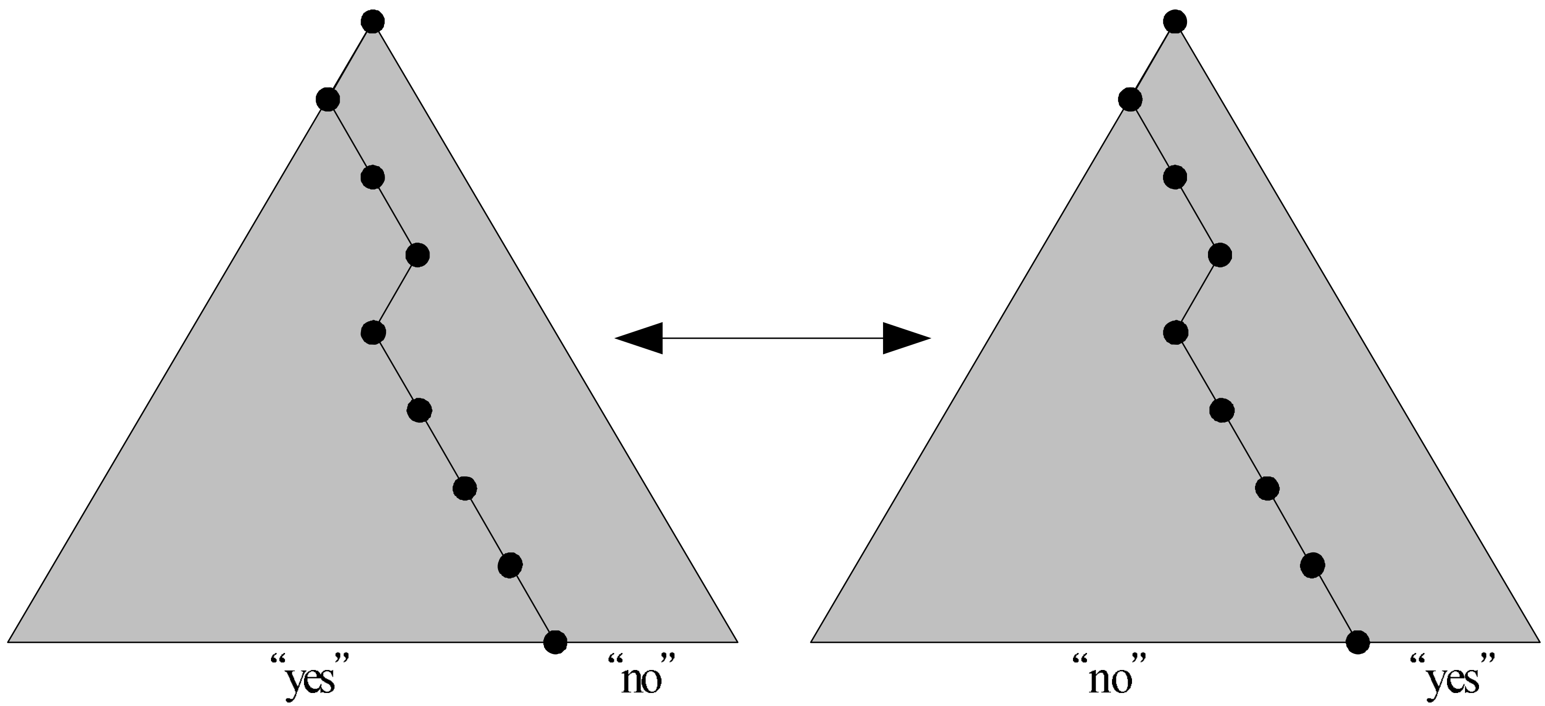
[a]See p. 641.

coBPP

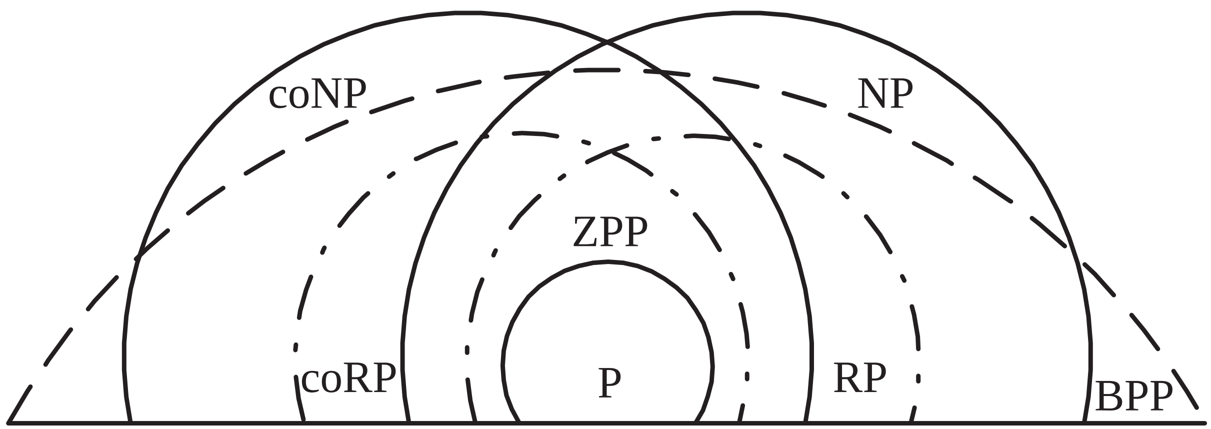
- The definition of BPP is symmetric: acceptance by clear majority and rejection by clear majority.
- An algorithm for $L \in$ BPP becomes one for $\bar{L}$ by reversing the answer.
- So $\bar{L} \in$ BPP and BPP $\subseteq$ coBPP.
- Similarly coBPP $\subseteq$ BPP.
- Hence BPP $=$ coBPP.
- This approach does not work for RP.[a]

[a] It did not work for NP either.

BPP and coBPP



"The Good, the Bad, and the Ugly"



Circuit Complexity

- Circuit complexity is based on boolean circuits instead of Turing machines.
- A boolean circuit with n inputs computes a boolean function of n variables.
- Now, identify `true`/1 with "yes" and `false`/0 with "no."
- Then a boolean circuit with n inputs accepts certain strings in $\{0, 1\}^n$.
- To relate circuits with an arbitrary language, we need one circuit for each possible input length n.

Formal Definitions

- The **size** of a circuit is the number of *gates* in it.
- A **family of circuits** is an infinite sequence $\mathcal{C} = (C_0, C_1, \ldots)$ of boolean circuits, where C_n has n boolean inputs.
- For input $x \in \{0,1\}^*$, $C_{|x|}$ outputs 1 if and only if $x \in L$.
- In other words,

$$C_n \text{ accepts } L \cap \{0,1\}^n.$$

Formal Definitions (concluded)

- $L \subseteq \{0, 1\}^*$ has **polynomial circuits** if there is a family of circuits $\mathcal{C}$ such that:
  - The size of C_n is at most $p(n)$ for some fixed polynomial p.
  - C_n accepts $L \cap \{0, 1\}^n$.

Exponential Circuits Suffice for All Languages

- Theorem 16 (p. 219) implies that there are languages that cannot be solved by circuits of size $2^n/(2n)$.
- But surprisingly, circuits of size 2^{n+2} can solve *all* problems, decidable or otherwise!

Exponential Circuits Suffice for All Languages (continued)

Proposition 79 *All decision problems (decidable or otherwise) can be solved by a circuit of size* 2^{n+2} *and depth* $2n$.

- We will show that for any language $L \subseteq \{0,1\}^*$, $L \cap \{0,1\}^n$ can be decided by a circuit of size 2^{n+2}.
- Define boolean function $f : \{0,1\}^n \to \{0,1\}$, where

$$f(x_1 x_2 \cdots x_n) = \begin{cases} 1, & x_1 x_2 \cdots x_n \in L, \\ 0, & x_1 x_2 \cdots x_n \notin L. \end{cases}$$

The Proof (concluded)

- Clearly, any circuit that implements f decides $L \cap \{ 0, 1 \}^n$.
- Now,

$$f(x_1 x_2 \cdots x_n) = (x_1 \wedge f(1x_2 \cdots x_n)) \vee (\neg x_1 \wedge f(0x_2 \cdots x_n)).$$

- The circuit size $s(n)$ for $f(x_1 x_2 \cdots x_n)$ hence satisfies

$$s(n) = 4 + 2s(n-1)$$

  with $s(1) = 1$.
- Solve it to obtain $s(n) = 5 \times 2^{n-1} - 4 \leq 2^{n+2}$.

The Circuit Complexity of P

Proposition 80 *All languages in P have polynomial circuits.*

- Let $L \in \mathrm{P}$ be decided by a TM in time $p(n)$.
- By Corollary 35 (p. 328), there is a circuit with $O(p(n)^2)$ gates that accepts $L \cap \{0, 1\}^n$.
- The size of that circuit depends only on L and the length of the input.
- The size of that circuit is polynomial in n.

Polynomial Circuits vs. P

- Is the converse of Proposition 80 true?
  - Do polynomial circuits accept only languages in P?
- No.
- Polynomial circuits can accept *undecidable* languages![a]

[a]See p. 268 of the textbook.

BPP's Circuit Complexity: Adleman's Theorem

Theorem 81 (Adleman, 1978) *All languages in BPP have polynomial circuits.*

- Our proof will be *nonconstructive* in that only the existence of the desired circuits is shown.
  - Recall our proof of Theorem 16 (p. 219).
  - Something exists if its probability of existence is nonzero.
- It is not known how to efficiently generate circuit C_n.
  - If the construction of C_n can be made efficient, then P = BPP, an unlikely result.

The Proof

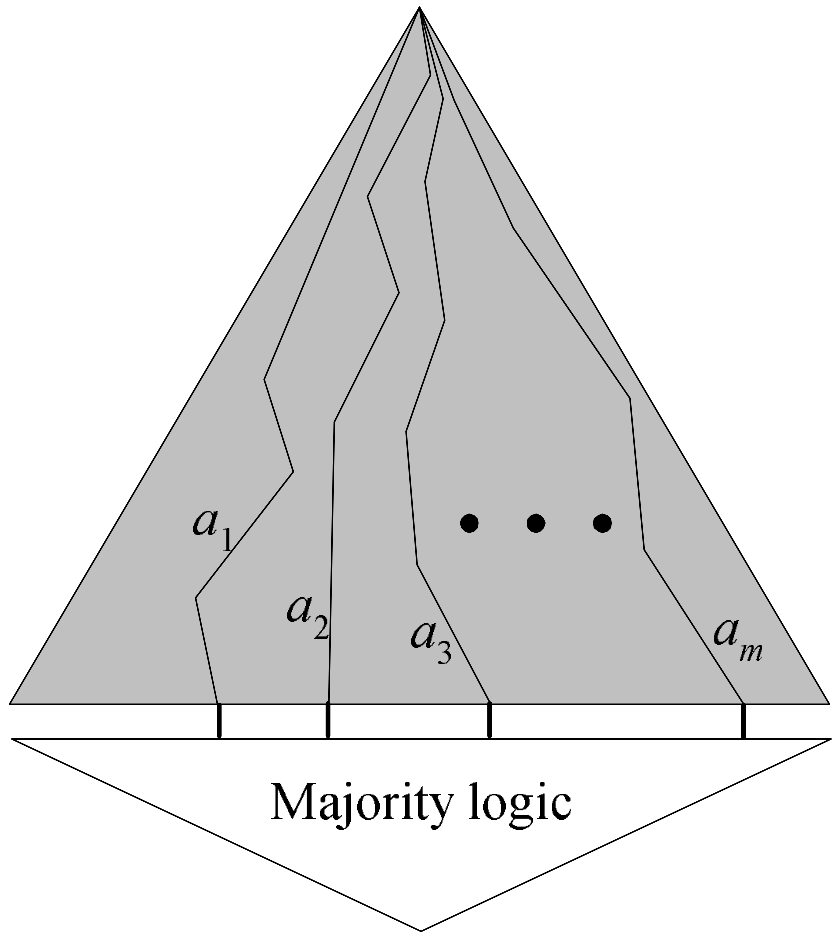
- Let $L \in \text{BPP}$ be decided by a precise polynomial-time NTM N by clear majority.
- We shall prove that L has polynomial circuits $C_0, C_1, \ldots$.
  - These *deterministic* circuits do not err.
- Suppose N runs in time $p(n)$, where $p(n)$ is a polynomial.
- Let $A_n = \{ a_1, a_2, \ldots, a_m \}$, where $a_i \in \{ 0, 1 \}^{p(n)}$.
- Each $a_i \in A_n$ represents a sequence of nondeterministic choices (i.e., a computation path) for N.
- Pick $m = 12(n + 1)$.

The Proof (continued)

- Let x be an input with $|x| = n$.
- Circuit C_n simulates N on x with all sequences of choices in A_n and then takes the majority of the m outcomes.[a]
  - Note that each A_n yields a circuit.
- As N with a_i is a polynomial-time deterministic TM, it can be simulated by polynomial circuits of size $O(p(n)^2)$.
  - See the proof of Proposition 80 (p. 639).

[a] As m is even, there may be no clear majority. Still, the probability of that happening is very small and does not materially affect our general conclusion. Thanks to a lively class discussion on December 14, 2010.

The Circuit



The Proof (continued)

- The size of C_n is therefore $O(mp(n)^2) = O(np(n)^2)$.
  - This is a polynomial.
- We now confirm the existence of an A_n making C_n correct on *all* n-bit inputs.
- Call a_i **bad** if it leads N to an error (a false positive or a false negative) for x.
- Select A_n uniformly randomly.

The Proof (continued)

- For each $x \in \{0, 1\}^n$, $1/4$ of the computations of N are erroneous.
- Because the sequences in A_n are chosen randomly and independently, the expected number of bad a_i's is $m/4$.[a]
- Also note after fixing the input x, the circuit is a function of the random bits.

[a] So the proof will not work for NP. Contributed by Mr. Ching-Hua Yu (D00921025) on December 11, 2012.

The Proof (continued)

- By the Chernoff bound (p. 618), the probability that the number of bad a_i's is $m/2$ or more is at most

$$e^{-m/12} = 2^{-(n+1)}.$$

- The error probability of using the majority rule is thus

$$\leq 2^{-(n+1)}$$

for each $x \in \{0, 1\}^n$.

The Proof (continued)

- The probability that there is an x such that A_n results in an incorrect answer is

$$\leq 2^n 2^{-(n+1)} = 2^{-1}$$

  by the union bound (**Boole's inequality**).[a]

- We just showed that at least half of the random A_n are correct.

- So with probability ≥ 0.5, a random A_n produces a correct C_n for *all* inputs of length n.
  - Of course, verifying this fact may take a long time.

[a] That is, $\text{prob}[\, A \cup B \cup \cdots \,] \leq \text{prob}[\, A \,] + \text{prob}[\, B \,] + \cdots$.

The Proof (concluded)

- Because this probability exceeds 0, an A_n that makes majority vote work for *all* inputs of length n exists.
- Hence a correct C_n exists.[a]
- We have used the **probabilistic method**[b] popularized by Erdős (1947).[c]
- This result answers the question on p. 549 with a "yes."

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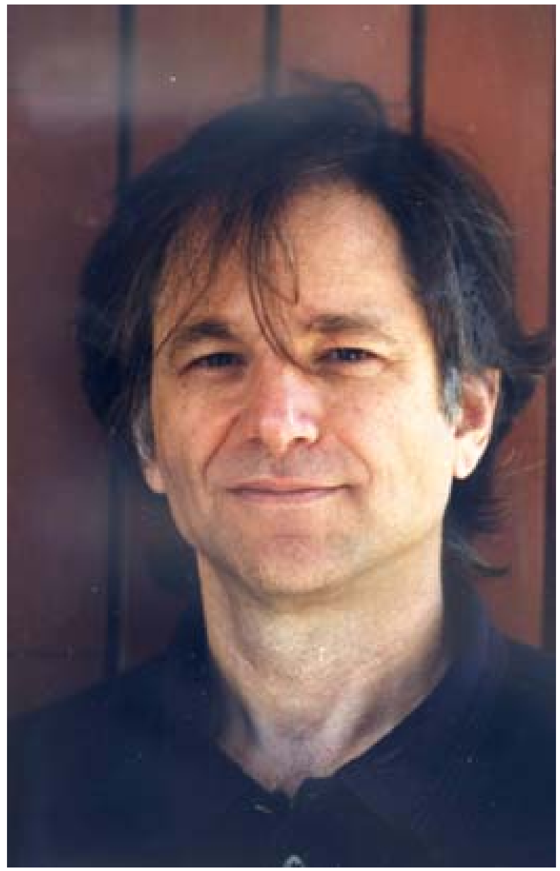
[a]Quine (1948), "To be is to be the value of a bound variable."
[b]A counting argument in the probabilistic language.
[c]Szele (1943) and Turán (1934) were earlier.

Leonard Adleman[a] (1945–)



[a]Turing Award (2002).

Paul Erdős (1913–1996)

Cryptography

Whoever wishes to keep a secret
must hide the fact that he possesses one.
— Johann Wolfgang von Goethe (1749–1832)

Cryptography

- **Alice** (A) wants to send a message to **Bob** (B) over a channel monitored by **Eve** (eavesdropper).
- The protocol should be such that the message is known only to Alice and Bob.
- The art and science of keeping messages secure is **cryptography**.

$$\text{Alice} \xrightarrow{\text{Eve}} \text{Bob}$$

Encryption and Decryption

- Alice and Bob agree on two algorithms E and D—the **encryption** and the **decryption algorithms**.
- Both E and D are known to the public in the analysis.
- Alice runs E and wants to send a message x to Bob.
- Bob operates D.

Encryption and Decryption (concluded)

- Privacy is assured in terms of two numbers e, d, the **encryption** and **decryption keys**.
- Alice sends $y = E(e, x)$ to Bob, who then performs $D(d, y) = x$ to recover x.
- x is called **plaintext**, and y is called **ciphertext**.[a]

[a]Both "zero" and "cipher" come from the same Arab word.

Some Requirements

- D should be an inverse of E given e and d.
- D and E must both run in (probabilistic) polynomial time.
- Eve should not be able to recover x from y without knowing d.
  - As D is public, d must be kept secret.
  - e may or may not be a secret.

Degree of Security

- **Perfect secrecy**: After a ciphertext is intercepted by the enemy, the a posteriori probabilities of the plaintext that this ciphertext represents are identical to the a priori probabilities of the same plaintext before the interception.
  - The probability that plaintext $\mathcal{P}$ occurs is independent of the ciphertext $\mathcal{C}$ being observed.
  - So knowing $\mathcal{C}$ yields no advantage in recovering $\mathcal{P}$.

Degree of Security (concluded)

- Such systems are said to be **informationally secure**.
- A system is **computationally secure** if breaking it is theoretically possible but computationally infeasible.

Conditions for Perfect Secrecy[a]

- Consider a cryptosystem where:
  - The space of ciphertext is as large as that of keys.
  - Every plaintext has a nonzero probability of being used.
- It is **perfectly secure** if and only if the following hold.
  - A key is chosen with uniform distribution.
  - For each plaintext x and ciphertext y, there exists a unique key e such that $E(e, x) = y$.

[a]Shannon (1949).

The One-Time Pad[a]

1: Alice generates a random string r as long as x;

2: Alice sends r to Bob over a secret channel;

3: Alice sends $x \oplus r$ to Bob over a public channel;

4: Bob receives y;

5: Bob recovers $x := y \oplus r$;

---

[a] Mauborgne & Vernam (1917); Shannon (1949). It was allegedly used for the hotline between Russia and the U.S.

Analysis

- The one-time pad uses $e = d = r$.
- This is said to be a **private-key cryptosystem**.
- Knowing x and knowing r are equivalent.
- Because r is random and private, the one-time pad achieves perfect secrecy.[a]
- The random bit string must be new for each round of communication.
- But the assumption of a private channel is problematic.

---

[a]See p. 660.

Chosen-Plaintext Attack

- Suppose Eve can obtain the ciphertexts for any plaintexts of her choice.
- She can ask the encryption algorithm to encrypt an arbitrary plaintext x to obtain cypertext y.
- Then she analyze those pairs to attack the cryptosystem.
- This is called the **chosen-plaintext attack**.
- For the one-time pad, she only has to perform $y \oplus x$ with a single pair (x, y) to recover the private key r!

Public-Key Cryptography[a]

- Suppose only d is private to Bob, whereas e is public knowledge.
- Bob generates the (e, d) pair and publishes e.
- Anybody like Alice can send $E(e, x)$ to Bob.
- Knowing d, Bob can recover x via

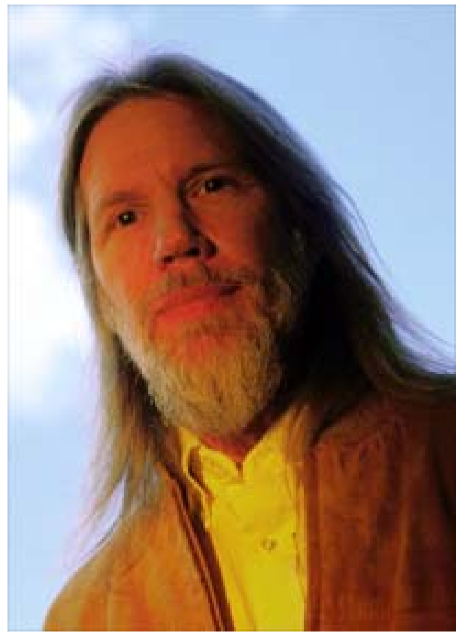
$$D(d, E(e, x)) = x.$$

[a] Diffie & Hellman (1976).

Public-Key Cryptography (concluded)

- The assumptions are complexity-theoretic.
  - It is computationally difficult to compute d from e.
  - It is computationally difficult to compute x from y without knowing d.

Whitfield Diffie[a] (1944–)



[a]Turing Award (2016).

Martin Hellman[a] (1945–)

[a]Turing Award (2016).

Complexity Issues

- Given y and x, it is easy to verify whether $E(e, x) = y$.
- Hence one can always guess an x and verify.
- Cracking a public-key cryptosystem is thus in NP.
- A *necessary* condition for the existence of secure public-key cryptosystems is P $\neq$ NP.
- But more is needed than P $\neq$ NP.
- For instance, it is not sufficient that D is hard to compute in the *worst* case.
- It should be hard in "most" or "average" cases.