Function Problems

- Decision problems are yes/no problems (SAT, TSP (D), etc.).
- **Function problems** require a solution (a satisfying truth assignment, a best TSP tour, etc.).
- Optimization problems are clearly function problems.
- What is the relation between function and decision problems?
- Which one is harder?

Function Problems Cannot Be Easier than Decision Problems

- If we know how to generate a solution, we can solve the corresponding decision problem.
  - If you can find a satisfying truth assignment efficiently, then SAT is in P.
  - If you can find the best TSP tour efficiently, then TSP (D) is in P.
- But we shall see that decision problems can be as hard as the corresponding function problems.

FSAT

- FSAT is this function problem:
  - Let $\phi(x_1, x_2, \ldots, x_n)$ be a boolean expression.
  - If ϕ is satisfiable, then return a satisfying truth assignment.
  - Otherwise, return "no."
- We next show that if SAT $\in$ P, then FSAT has a polynomial-time algorithm.
- SAT is a subroutine (black box) that returns "yes" or "no" on the satisfiability of the input.

An Algorithm for FSAT Using SAT

```
t := ε; {Truth assignment.}
if φ ∈ SAT then
    for i = 1, 2, ..., n do
        if φ[ x_i = true ] ∈ SAT then
            t := t ∪ { x_i = true };
            φ := φ[ x_i = true ];
        else
            t := t ∪ { x_i = false };
            φ := φ[ x_i = false ];
        end if
    end for
    return t;
else
    return "no";
end if
```


Analysis

- If SAT can be solved in polynomial time, so can FSAT.
  - There are $\leq n+1$ calls to the algorithm for SAT.[a]
  - Boolean expressions shorter than ϕ are used in each call to the algorithm for SAT.
- Hence SAT and FSAT are equally hard (or easy).

---

[a]Contributed by Ms. Eva Ou (R93922132) on November 24, 2004.

Analysis (concluded)

- Note that this reduction from FSAT to SAT is not a Karp reduction.[a]
  - Will the set of NP-complete problems differ under different reductions?[b]
- Instead, it calls SAT multiple times as a subroutine, and its answers guide the search on the computation tree.

---

[a] Recall p. 273 and p. 278.

[b] Contributed by Mr. Yu-Ming Lu (R06723032, D08922008) and Mr. Han-Ting Chen (R10922073) on December 9, 2021.

TSP and TSP (D) Revisited

- We are given n cities $1, 2, \ldots, n$ and integer distances $d_{ij} = d_{ji}$ between any two cities i and j.
- TSP (D) asks if there is a tour with a total distance at most B.
- TSP asks for a tour with the shortest total distance.
  - The shortest total distance is at most $\sum_{i,j} d_{ij}$.
    * Recall that the input string contains $d_{11}, \ldots, d_{nn}$.
- Thus the shortest total distance is less than $2^{|\,x\,|}$ in magnitude, where x is the input (why?).
- We next show that if TSP (D) $\in$ P, then TSP has a polynomial-time algorithm.

An Algorithm for TSP Using TSP (D)

```
Perform a binary search over interval [0, 2^{|x|}] by calling
    TSP (D) to obtain the shortest distance, C;
for i, j = 1, 2, ..., n do
    Call TSP (D) with B = C and d_{ij} = C + 1;
    if "no" then
        Restore d_{ij} to its old value; {Edge [i, j] is critical.}
    end if
end for
return the tour with edges whose d_{ij} ≤ C;
```

Analysis

- An edge which is not on *any* remaining optimal tours will be eliminated, with its d_{ij} set to $C + 1$.
- So the algorithm ends with n edges which are not eliminated (why?).
- This is true even if there are multiple optimal tours![a]

---

[a] Thanks to a lively class discussion on November 12, 2013 and December 9, 2021.

Analysis (concluded)

- There are $O(|x| + n^2)$ calls to the algorithm for TSP (D).
- Each call has an input length of $O(|x|)$.
- So if TSP (D) can be solved in polynomial time, so can TSP.
- Hence TSP (D) and TSP are equally hard (or easy).[a]

[a]How about counting the number of optimal TSP tours? This is related to #P-completeness (p. 867). Contributed by Mr. Vincent Hwang (R10922138) on December 9, 2021.

Randomized Computation

I know that half my advertising works,
I just don't know which half.
— John Wanamaker

I know that half my advertising is
a waste of money,
I just don't know which half!
— McGraw-Hill ad.

Randomized Algorithms[a]

- Randomized algorithms flip unbiased coins.
- There are important problems for which there are no known efficient *deterministic* algorithms but for which very efficient randomized algorithms exist.
  - Extraction of square roots, for instance.
- There are problems where randomization is *necessary*.
  - Secure protocols.
- Randomized version can be more efficient.
  - Parallel algorithms for maximal independent set.[b]

[a] Rabin (1976); Solovay & Strassen (1977).
[b] "Maximal" (a local maximum) not "maximum" (a global maximum).

Randomized Algorithms (concluded)

- Are randomized algorithms algorithms?[a]
- Coin flips are occasionally used in politics.[b]

---

[a]Pascal, "Truth is so delicate that one has only to depart the least bit from it to fall into error."

[b]In the 2016 Iowa Democratic caucuses, e.g. (see `http://edition.cnn.com/2016/02/02/politics/hillary-clinton-coin-flip-iowa-bernie-sanders/index.html`).

"Four Most Important Randomized Algorithms"[a]

1. Primality testing.[b]
2. Graph connectivity using random walks.[c]
3. Polynomial identity testing.[d]
4. Algorithms for approximate counting.[e]

---

[a] Trevisan (2006).

[b] Rabin (1976); Solovay & Strassen (1977).

[c] Aleliunas, Karp, Lipton, Lovász, & Rackoff (1979).

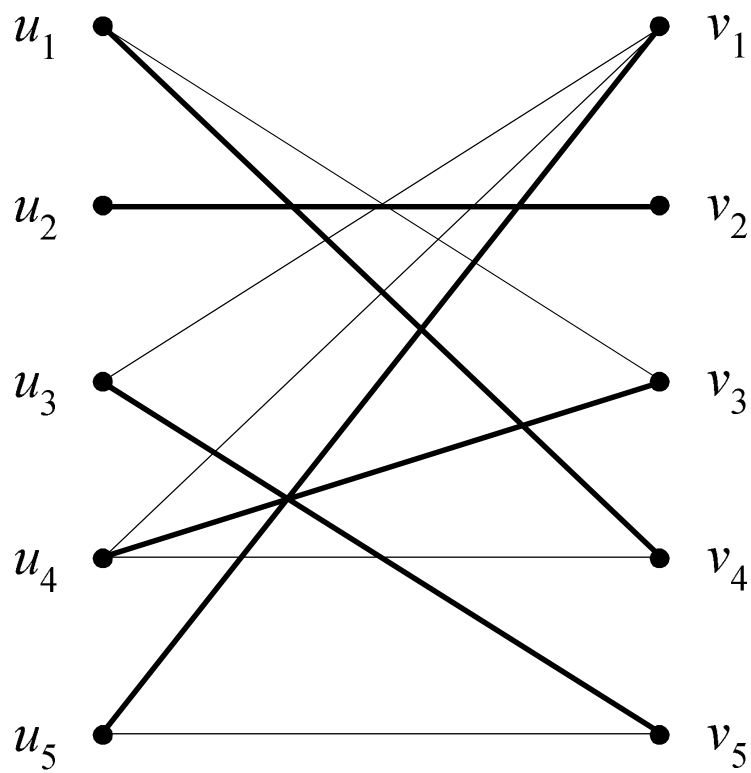
[d] Schwartz (1980); Zippel (1979).

[e] Sinclair & Jerrum (1989).

Bipartite Perfect Matching

- We are given a **bipartite graph** $G = (U, V, E)$.
  - $U = \{ u_1, u_2, \ldots, u_n \}$.
  - $V = \{ v_1, v_2, \ldots, v_n \}$.
  - $E \subseteq U \times V$.
- We are asked if there is a **perfect matching**.
  - A permutation π of $\{ 1, 2, \ldots, n \}$ such that
  $$(u_i, v_{\pi(i)}) \in E$$
  for all $i \in \{ 1, 2, \ldots, n \}$.
- A perfect matching contains n edges.

A Perfect Matching in a Bipartite Graph



Symbolic Determinants

- We are given a bipartite graph G.
- Construct the $n \times n$ matrix A^G whose (i, j)th entry A^G_{ij} is a symbolic variable x_{ij} if $(u_i, v_j) \in E$ and 0 otherwise:

$$A^G_{ij} = \begin{cases} x_{ij}, & \text{if } (u_i, v_j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Symbolic Determinants (continued)

- The matrix for the bipartite graph G on p. 532 is[a]

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & x_{14} & 0 \\ 0 & x_{22} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & x_{35} \\ x_{41} & 0 & x_{43} & x_{44} & 0 \\ x_{51} & 0 & 0 & 0 & x_{55} \end{bmatrix}. \tag{8}$$

[a]The idea is similar to the Tanner (1981) graph in coding theory.

Symbolic Determinants (concluded)

- The **determinant** of A^G is

$$\det(A^G) = \sum_{\pi} \mathrm{sgn}(\pi) \prod_{i=1}^{n} A^G_{i,\pi(i)}. \tag{9}$$

  - π ranges over all permutations of n elements.
  - $\mathrm{sgn}(\pi)$ is 1 if π is the product of an even number of transpositions and -1 otherwise.[a]
- $\det(A^G)$ contains $n!$ terms, many of which may be 0s.

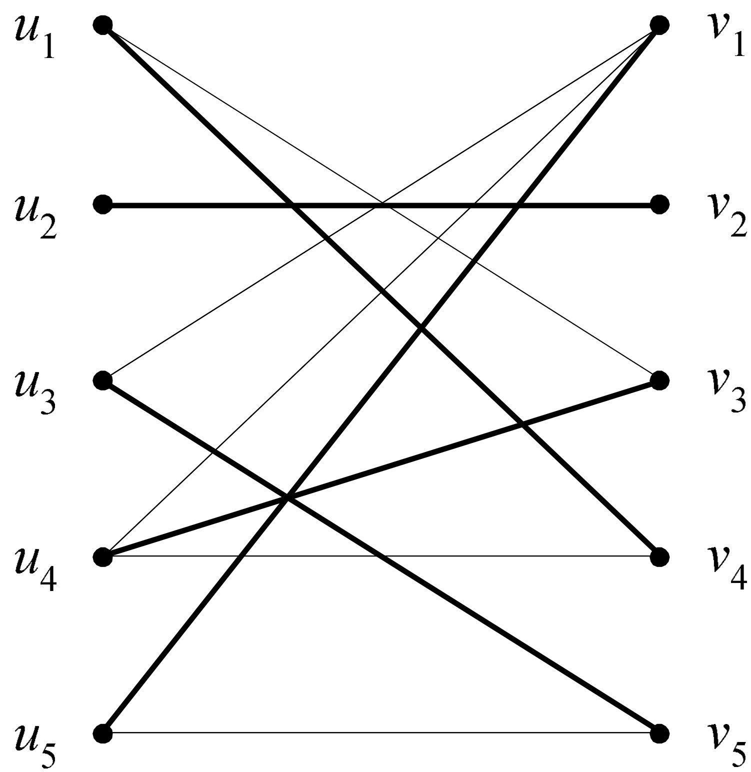
[a] Equivalently, $\mathrm{sgn}(\pi) = 1$ if the number of (i, j)s such that $i < j$ and $\pi(i) > \pi(j)$ is even. Contributed by Mr. Hwan-Jeu Yu (`D95922028`) on May 1, 2008.

Determinant and Bipartite Perfect Matching

- In $\sum_\pi \mathrm{sgn}(\pi) \prod_{i=1}^n A^G_{i,\pi(i)}$, note the following:
  - Each summand corresponds to a possible perfect matching π.
  - Nonzero summands $\prod_{i=1}^n A^G_{i,\pi(i)}$ are distinct monomials and *will not cancel*.
- $\det(A^G)$ is essentially an exhaustive enumeration.

Proposition 65 (Edmonds, 1967) *G has a perfect matching if and only if $\det(A^G)$ is not identically zero.*

Perfect Matching and Determinant (p. 532)



Perfect Matching and Determinant (concluded)

- The matrix is (p. 534)

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & \boxed{x_{14}} & 0 \\ 0 & \boxed{x_{22}} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & \boxed{x_{35}} \\ x_{41} & 0 & \boxed{x_{43}} & x_{44} & 0 \\ \boxed{x_{51}} & 0 & 0 & 0 & x_{55} \end{bmatrix}.$$

- $\det(A^G) = -x_{14}x_{22}x_{35}x_{43}x_{51} + x_{13}x_{22}x_{35}x_{44}x_{51} + x_{14}x_{22}x_{31}x_{43}x_{55} - x_{13}x_{22}x_{31}x_{44}x_{55}$.
- Each nonzero term denotes a perfect matching, and vice versa.

How To Test If a Polynomial Is Identically Zero?

- $\det(A^G)$ is a polynomial in n^2 variables.
- It has, potentially, exponentially many terms.
- Expanding the determinant polynomial is thus infeasible.
- If $\det(A^G) \equiv 0$, then it remains zero if we substitute *arbitrary* integers for the variables $x_{11}, \ldots, x_{nn}$.
- When $\det(A^G) \not\equiv 0$, what is the likelihood of obtaining a zero?

Number of Roots of a Polynomial

Lemma 66 (Schwartz, 1980) *Let* $p(x_1, x_2, \ldots, x_m) \not\equiv 0$ *be a polynomial in* m *variables each of degree at most* d. *Let* $M \in \mathbb{Z}^+$. *Then the number of* m*-tuples*

$$(x_1, x_2, \ldots, x_m) \in \{\, 0, 1, \ldots, M-1 \,\}^m$$

such that $p(x_1, x_2, \ldots, x_m) = 0$ *is*

$$\leq mdM^{m-1}.$$

- By induction on m (consult the textbook).

Density Attack

- The density of roots in the domain is at most
$$\frac{mdM^{m-1}}{M^m} = \frac{md}{M}. \tag{10}$$
- So suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
- Then a random
$$(x_1, x_2, \ldots, x_m) \in \{\, 0, 1, \ldots, M-1 \,\}^m$$
has a probability of $\leq md/M$ of being a root of p.
- Note that M is under our control!
  - One can raise M to lower the error probability, e.g.

Density Attack (concluded)

Here is a sampling algorithm to test if $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.

```
Choose i_1, ..., i_m from { 0, 1, ..., M − 1 } randomly;
if p(i_1, i_2, ..., i_m) ≠ 0 then
    return "p is not identically zero";
else
    return "p is (probably) identically zero";
end if
```

Analysis

- If $p(x_1, x_2, \ldots, x_m) \equiv 0$, the algorithm will always be correct as $p(i_1, i_2, \ldots, i_m) = 0$.
- Suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
  - The algorithm will answer incorrectly with probability at most md/M by Eq. (10) on p. 541.
- We next return to the original problem of bipartite perfect matching.

A Randomized Bipartite Perfect Matching Algorithm[a]

1: Choose n^2 integers $i_{11}, \ldots, i_{nn}$ from $\{ 0, 1, \ldots, 2n^2 - 1 \}$ randomly; {So $M = 2n^2$.}
2: Calculate $\det(A^G(i_{11}, \ldots, i_{nn}))$ by Gaussian elimination;
3: **if** $\det(A^G(i_{11}, \ldots, i_{nn})) \neq 0$ **then**
4: **return** "G has a perfect matching";
5: **else**
6: **return** "G has (probably) no perfect matchings";
7: **end if**

---

[a]Lovász (1979). According to Paul Erdős, Lovász wrote his first significant paper "at the ripe old age of 17."

Analysis

- If G has no perfect matchings, the algorithm will always be correct as $\det(A^G(i_{11}, \ldots, i_{nn})) = 0$.
- Suppose G has a perfect matching.
  - The algorithm will answer incorrectly with probability at most $md/M = 0.5$ with $m = n^2$, $d = 1$ and $M = 2n^2$ in Eq. (10) on p. 541.
- Run the algorithm *independently* k times.
- Output "G has no perfect matchings" if and only if *all* say "(probably) no perfect matchings."
- The error probability is now reduced to at most 2^{-k}.

Lószló Lovász (1948–)

Remarks[a]

- Note that we calculated

$$\text{prob}[\,\text{algorithm answers "no"}\mid G \text{ has no perfect matchings}\,],$$

$$\text{prob}[\,\text{algorithm answers "yes"}\mid G \text{ has a perfect matching}\,].$$

  - And they are 1 and $\geq 1/2$, respectively.

- We did *not* calculate[b]

$$\text{prob}[\,G \text{ has no perfect matchings}\mid \text{algorithm answers "no"}\,],$$

$$\text{prob}[\,G \text{ has a perfect matching}\mid \text{algorithm answers "yes"}\,].$$

---

[a] Thanks to a lively class discussion on May 1, 2008.

[b] *Numerical Recipes in C* (1988), "statistics is *not* a branch of mathematics!" Similar issues arise in MAP (maximum a posteriori) estimates and ML (maximum likelihood) estimates.

But How Large Can det$(A^G(i_{11}, \ldots, i_{nn}))$ Be?

- It is at most[a]

$$n!\left(2n^2\right)^n.$$

- Stirling's formula says $n! \sim \sqrt{2\pi n}\,(n/e)^n$.
- Hence

$$\log_2 \det(A^G(i_{11}, \ldots, i_{nn})) = O(n \log_2 n)$$

bits are sufficient for representing the determinant.

- We skip the details about how to make sure that all *intermediate* results are of polynomial size.

---

[a]In fact, it can be lowered to $2^{O(\log^2 n)}$ (Csanky, 1975)!

An Intriguing Question[a]

- Is there an $(i_{11}, \ldots, i_{nn})$ that will always give correct answers for the algorithm on p. 544?
- A theorem on p. 641 shows that such an $(i_{11}, \ldots, i_{nn})$ exists!
  - Whether it can be found efficiently is another matter.
- Once $(i_{11}, \ldots, i_{nn})$ is available, the algorithm can be made deterministic.
  - Is it an algorithm for bipartite perfect matching?[b]

---

[a]Thanks to a lively class discussion on November 24, 2004.

[b]We have one algorithm for each n — unless there is an algorithm to generate such $(i_{11}, \ldots, i_{nn})$ for all n. Contributed by Mr. Han-Ting Chen (`R10922073`) on December 9, 2021.

Randomization vs. Nondeterminism[a]

- What are the differences between randomized algorithms and nondeterministic algorithms?
- Think of a randomized algorithm as a nondeterministic one but with a probability associated with every guess/branch.
- So each computation path of a randomized algorithm has a probability associated with it.

[a] Contributed by Mr. Olivier Valery (D01922033) and Mr. Hasan Alhasan (D01922034) on November 27, 2012.

Monte Carlo Algorithms[a]

- The randomized bipartite perfect matching algorithm is called a **Monte Carlo algorithm** in the sense that
  - If the algorithm finds that a matching exists, it is always correct (no **false positives**; no **type I errors**).
  - If the algorithm answers in the negative, then it may make an error (**false negatives**; **type II errors**).
    - And the error probability must be small.

[a]Metropolis & Ulam (1949).

Monte Carlo Algorithms (continued)

- The algorithm makes a false negative with probability ≤ 0.5.[a]
- Again, this probability refers to[b]

$$\text{prob}[\,\text{algorithm answers "no"} \mid G \text{ has a perfect matching}\,]$$

not

$$\text{prob}[\,G \text{ has a perfect matching} \mid \text{algorithm answers "no"}\,].$$

---

[a] Equivalently, among the coin flip sequences, at most half of them lead to the wrong answer.

[b] In general, prob[algorithm answers "no" | input is a yes instance].

Monte Carlo Algorithms (concluded)

- This probability 0.5 is *not* over the space of all graphs or determinants, but *over* the algorithm's own coin flips.
  - It holds for *any* bipartite graph.
- In contrast, to calculate

  $$\text{prob}[\, G \text{ has a perfect matching} \,|\, \text{algorithm answers "no"} \,],$$

  we will need the distribution of G.
- But it is an empirical statement that is very hard to verify.

The Markov Inequality[a]

Lemma 67 *Let x be a random variable taking nonnegative integer values. Then for any $k > 0$,*

$$\text{prob}[\, x \geq kE[\, x\,]\,] \leq 1/k.$$

- Let p_i denote the probability that $x = i$.

$$\begin{aligned} E[\, x\,] &= \sum_i i p_i = \sum_{i < kE[\, x\,]} i p_i + \sum_{i \geq kE[\, x\,]} i p_i \\ &\geq \sum_{i \geq kE[\, x\,]} i p_i \geq kE[\, x\,] \sum_{i \geq kE[\, x\,]} p_i \\ &\geq kE[\, x\,] \times \text{prob}[x \geq kE[\, x\,]]. \end{aligned}$$

[a]Andrei Andreyevich Markov (1856–1922).

Andrei Andreyevich Markov (1856–1922)

FSAT for k-SAT Formulas (p. 518)

- Let $\phi(x_1, x_2, \ldots, x_n)$ be a k-SAT formula.
- If ϕ is satisfiable, then return a satisfying truth assignment.
- Otherwise, return "no."
- We next propose a randomized algorithm for this problem.

A Random Walk Algorithm for ϕ in CNF Form

```
Start with an arbitrary truth assignment T;
for i = 1, 2, ..., r do
    if T ⊨ φ then
        return "φ is satisfiable with T";
    else
        Let c be an unsatisfied clause in φ under T; {All of
        its literals are false under T.}
        Pick any x of these literals at random;
        Modify T to make x true;
    end if
end for
return "φ is unsatisfiable";
```

3SAT vs. 2SAT Again

- Note that if ϕ is unsatisfiable, the algorithm will answer "unsatisfiable."
- The random walk algorithm needs expected exponential time for 3SAT.
  - In fact, it runs in expected $O((1.333 \cdots + \epsilon)^n)$ time with $r = 3n$,[a] much better than $O(2^n)$.[b]
- We will show immediately that it works well for 2SAT.
- The state of the art as of 2014 is expected $O(1.30704^n)$ time for 3SAT and expected $O(1.46899^n)$ time for 4SAT.[c]

---

[a]Use this setting per run of the algorithm.

[b]Schöning (1999). Makino, Tamaki, & Yamamoto (2011) improve the bound to deterministic $O(1.3303^n)$.

[c]Hertli (2014).

Random Walk Works for 2SAT[a]

Theorem 68 *Suppose the random walk algorithm with $r = 2n^2$ is applied to any satisfiable* 2SAT *problem with n variables. Then a satisfying truth assignment will be discovered with probability at least 0.5.*

- Let $\hat{T}$ be a truth assignment such that $\hat{T} \models \phi$.
- Assume our starting T differs from $\hat{T}$ in i values.
  - Their Hamming distance is i.
- Recall T is arbitrary.

[a]Papadimitriou (1991).

The Proof

- Let $t(i)$ denote the expected number of repetitions of the flipping step[a] until a satisfying truth assignment is found.
- It can be shown that $t(i)$ is finite.
- $t(0) = 0$ because it means that $T = \hat{T}$ and hence $T \models \phi$.
- If $T \neq \hat{T}$ or any other satisfying truth assignment, then we need to flip the coin at least once.
- We flip a coin to pick among the 2 literals of a clause not satisfied by the present T.
- At least one of the 2 literals is true under $\hat{T}$ because $\hat{T}$ satisfies all clauses.

[a] That is, Statement 7.

The Proof (continued)

- So we have at least a 50% chance of moving closer to $\hat{T}$.
- Thus
$$t(i) \leq \frac{t(i-1)+t(i+1)}{2} + 1$$
for $0 < i < n$.
  - Inequality is used because, for example, T may differ from $\hat{T}$ in both literals.
- It must also hold that
$$t(n) \leq t(n-1) + 1$$
because at $i = n$, we can only decrease i.

The Proof (continued)

- Now, put the necessary relations together:

$$t(0) = 0, \tag{11}$$

$$t(i) \leq \frac{t(i-1)+t(i+1)}{2} + 1, \quad 0 < i < n, \tag{12}$$

$$t(n) \leq t(n-1)+1. \tag{13}$$

- Technically, this is a one-dimensional random walk with an absorbing barrier at $i = 0$ and a reflecting barrier at $i = n$ (if we replace "$\leq$" with "$=$").[a]

[a] The proof in the textbook does exactly that. But a student pointed out difficulties with this proof technique on December 8, 2004. So our proof here uses the original inequalities.

The Proof (continued)

- Add up the relations for $2t(1), 2t(2), 2t(3), \ldots, 2t(n-1), t(n)$ to obtain[a]

$$\begin{aligned} & 2t(1) + 2t(2) + \cdots + 2t(n-1) + t(n) \\ \leq\; & t(0) + t(1) + 2t(2) + \cdots + 2t(n-2) + 2t(n-1) + t(n) \\ & +2(n-1) + 1. \end{aligned}$$

- Simplify it to yield

$$t(1) \leq 2n - 1. \tag{14}$$

[a] Adding up the relations for $t(1), t(2), t(3), \ldots, t(n-1)$ will also work, thanks to Mr. Yen-Wu Ti (D91922010).

The Proof (continued)

- Add up the relations for $2t(2), 2t(3), \ldots, 2t(n-1), t(n)$ to obtain

$$
\begin{aligned}
& 2t(2) + \cdots + 2t(n-1) + t(n) \\
\leq\ & t(1) + t(2) + 2t(3) + \cdots + 2t(n-2) + 2t(n-1) + t(n) \\
& +2(n-2) + 1.
\end{aligned}
$$

- Simplify it to yield

$$t(2) \leq t(1) + 2n - 3 \leq 2n - 1 + 2n - 3 = 4n - 4$$

by Eq. (14) on p. 563.

The Proof (continued)

- Continuing the process, we shall obtain[a]

$$t(i) \leq 2in - i^2.$$

- The worst upper bound happens when $i = n$, in which case

$$t(n) \leq n^2.$$

- We conclude that

$$t(i) \leq t(n) \leq n^2$$

for $0 \leq i \leq n$.

---

[a]See also Feller (1968).

The Proof (concluded)

- So the expected number of steps is at most n^2.
- The algorithm picks $r = 2n^2$.
- Apply the Markov inequality (p. 554) with $k = 2$ to yield the desired probability of 0.5.
- The proof does *not* yield a polynomial bound for 3SAT.[a]

[a] Contributed by Mr. Cheng-Yu Lee (R95922035) on November 8, 2006.

Boosting the Performance

- We can pick $r = 2mn^2$ to have an error probability of
$$\leq \frac{1}{2m}$$
by Markov's inequality.
- Alternatively, with the same running time, we can run the "$r = 2n^2$" algorithm m times.
- The error probability is now reduced to
$$\leq 2^{-m}.$$

Primality Tests

- PRIMES asks if a number N is a prime.
- The classic algorithm tests if $k \mid N$ for $k = 2, 3, \ldots, \sqrt{N}$.
- But it runs in $\Omega(2^{(\log_2 N)/2})$ steps.
- COMPOSITENESS asks if a number is composite.

The Fermat Test for Primality

Fermat's "little" theorem (p. 504) suggests the following primality test for any given number N:

```
Pick a number a randomly from { 1, 2, ..., N − 1 };
if a^(N−1) ≢ 1 mod N then
    return "N is composite";
else
    return "N is (probably) a prime";
end if
```

The Fermat Test for Primality (continued)

- **Carmichael numbers** are composite numbers that will pass the Fermat test for *all* $a \in \{1, 2, \ldots, N-1\}$.[a]
  - The Fermat test will return "N is a prime" for all Carmichael numbers N.
- If there are finitely many Carmichael numbers, store them for matches before running the Fermat test.
- Unfortunately, there are infinitely many such numbers.[b]
  - The number of Carmichael numbers less than N exceeds $N^{2/7}$ for N large enough.

---

[a] Carmichael (1910). Lo (1994) mentions an investment strategy based on such numbers!

[b] Alford, Granville, & Pomerance (1992).

The Fermat Test for Primality (concluded)

- The Fermat test will fail all of them.
- So the Fermat test is an *incorrect* algorithm for PRIMES.
- Even suppose N is not a Carmichael number but remains composite.
- We need many $a \in \{ 1, 2, \ldots, N-1 \}$ such that $a^{N-1} \not\equiv 1 \bmod N$.
- Otherwise, the correct answer will come only with a vanishing probability (say $1/N$).[a]

[a] Contributed by Mr. Vincent Hwang (R10922138) on December 9, 2021.

Square Roots Modulo a Prime

- Equation $x^2 \equiv a \bmod p$ has at most two (distinct) roots by Lemma 64 (p. 509).
  - The roots are called **square roots**.
  - Numbers a with square roots *and* $\gcd(a, p) = 1$ are called **quadratic residues**.
    - They are

$$1^2 \bmod p, 2^2 \bmod p, \ldots, (p-1)^2 \bmod p.$$

- We shall show that a number either has two roots or has none, and testing which is the case is trivial.[a]

[a]But no efficient *deterministic* general-purpose square-root-extracting algorithms are known yet.

Euler's Test

Lemma 69 (Euler) *Let p be an odd prime and $a \neq 0 \bmod p$.*

1. *If*
$$a^{(p-1)/2} \equiv 1 \bmod p,$$
then $x^2 \equiv a \bmod p$ has two roots.

2. *If*
$$a^{(p-1)/2} \not\equiv 1 \bmod p,$$
then
$$a^{(p-1)/2} \equiv -1 \bmod p$$
and $x^2 \equiv a \bmod p$ has no roots.

The Proof (continued)

- Let r be a primitive root of p.
- Fermat's "little" theorem says $r^{p-1} \equiv 1 \bmod p$, so

$$r^{(p-1)/2}$$

  is a square root of 1.
- In particular,

$$r^{(p-1)/2} \equiv 1 \text{ or } -1 \bmod p.$$

- But as r is a primitive root, $r^{(p-1)/2} \not\equiv 1 \bmod p$.
- Hence $r^{(p-1)/2} \equiv -1 \bmod p$.

The Proof (continued)

- Let $a \equiv r^k \bmod p$ for some k.
- Suppose $a^{(p-1)/2} \equiv 1 \bmod p$.
- Then

$$1 \equiv a^{(p-1)/2} \equiv r^{k(p-1)/2} \equiv \left[r^{(p-1)/2} \right]^k \equiv (-1)^k \bmod p.$$

- So k must be even.

The Proof (continued)

- Suppose $a \equiv r^{2j} \bmod p$ for some $1 \leq j \leq (p-1)/2$.
- Then
$$a^{(p-1)/2} \equiv r^{j(p-1)} \equiv 1 \bmod p.$$
- The two *distinct* roots of a are
$$r^j, -r^j (\equiv r^{j+(p-1)/2} \bmod p).$$
  - If $r^j \equiv -r^j \bmod p$, then $2r^j \equiv 0 \bmod p$, which implies $r^j \equiv 0 \bmod p$, a contradiction as r is a primitive root.

The Proof (continued)

- As $1 \leq j \leq (p-1)/2$, there are $(p-1)/2$ such a's.
- Each such $a \equiv r^{2j} \bmod p$ has 2 distinct square roots.
- The square roots of all these a's are distinct.
  - The square roots of *different* a's must be different.
- Hence the set of *square roots* is $\{1, 2, \ldots, p-1\}$.
- As a result,

$$a = r^{2j} \bmod p, 1 \leq j \leq (p-1)/2,$$

exhaust all the quadratic residues.

The Proof (concluded)

- Suppose $a = r^{2j+1} \bmod p$ now.
- Then it has no square roots because all the square roots have been taken.
- Finally,

$$a^{(p-1)/2} \equiv \left[r^{(p-1)/2} \right]^{2j+1} \equiv (-1)^{2j+1} \equiv -1 \bmod p.$$

The Legendre Symbol[a] and Quadratic Residuacity Test

- By Lemma 69 (p. 573),

$$a^{(p-1)/2} \equiv \pm 1 \bmod p$$

for $a \not\equiv 0 \bmod p$.

- For odd prime p, define the **Legendre symbol** $(a \mid p)$ as

$$(a \mid p) \triangleq \begin{cases} 0, & \text{if } p \mid a, \\ 1, & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } a \text{ is a } \textbf{quadratic nonresidue} \text{ modulo } p. \end{cases}$$

- It is sometimes pronounced "a over p."

[a] Andrien-Marie Legendre (1752–1833).

The Legendre Symbol and Quadratic Residuacity Test (concluded)

- Euler's test (p. 573) implies

$$a^{(p-1)/2} \equiv (a \,|\, p) \bmod p$$

  for any odd prime p and any integer a.

- Note that $(ab \,|\, p) = (a \,|\, p)(b \,|\, p)$.

Gauss's Lemma

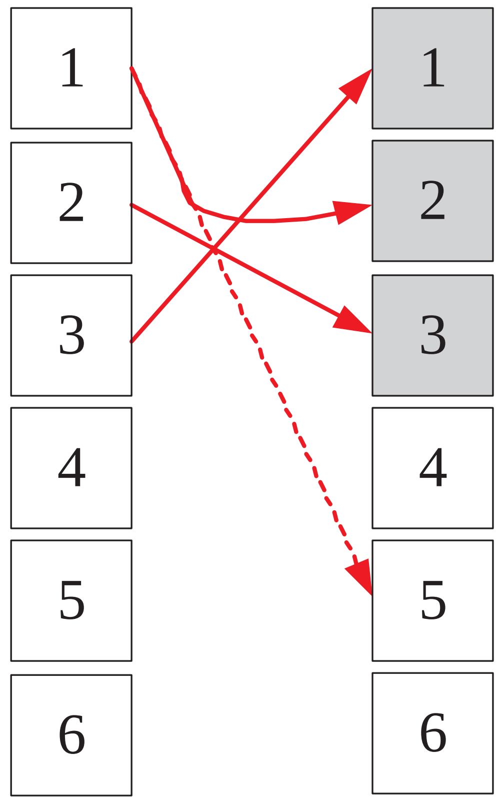
Lemma 70 (Gauss) *Let p and q be two distinct odd primes. Then $(q \,|\, p) = (-1)^m$, where m is the number of residues in $R \stackrel{\Delta}{=} \{\, iq \bmod p : 1 \leq i \leq (p-1)/2 \,\}$ that are greater than $(p-1)/2$.*

- All residues in R are distinct.
  - If $iq = jq \bmod p$, then $p \,|\, (j - i)$ or $p \,|\, q$.
  - But neither is possible.
- No two elements of R add up to p.
  - If $iq + jq \equiv 0 \bmod p$, then $p \,|\, (i + j)$ or $p \,|\, q$.
  - But neither is possible.

The Proof (continued)

- Replace each of the m elements $a \in R$ such that $a > (p-1)/2$ by $p - a$.
  - This is equivalent to performing $-a \bmod p$.
- Call the resulting set of residues R'.
- All numbers in R' are at most $(p-1)/2$.
- In fact, $R' = \{1, 2, \ldots, (p-1)/2\}$ (see illustration next page).
  - Otherwise, two elements of R would add up to p,[a] which has been shown to be impossible.

[a]Because then $iq \equiv -jq \bmod p$ for some $i \neq j$.



$p = 7$ and $q = 5$.

The Proof (concluded)

- Alternatively, $R' = \{\, \pm iq \bmod p : 1 \leq i \leq (p-1)/2 \,\}$, where exactly m of the elements have the minus sign.
- Take the product of all elements in the two representations of R'.
- So

$$[(p-1)/2]! \equiv (-1)^m q^{(p-1)/2} [(p-1)/2]! \bmod p.$$

- Because $\gcd([(p-1)/2]!, p) = 1$, the above implies

$$1 = (-1)^m q^{(p-1)/2} \bmod p.$$

Legendre's Law of Quadratic Reciprocity[a]

- Let p and q be two distinct odd primes.
- The next result says $(p \mid q)$ and $(q \mid p)$ are distinct if and only if both p and q are 3 mod 4.

Lemma 71 (Legendre, 1785; Gauss)

$$(p \mid q)(q \mid p) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

---

[a] First stated by Euler in 1751. Legendre (1785) did not give a correct proof. Gauss proved the theorem when he was 19. He gave at least 8 different proofs during his life. The 152nd proof appeared in 1963. A computer-generated formal proof was given in Russinoff (1990). As of 2008, there had been 4 such proofs. Wiedijk (2008), "the Law of Quadratic Reciprocity is the first nontrivial theorem that a student encounters in the mathematics curriculum."

The Proof (continued)

- Sum the elements of R' on p. 584 in mod2.
- On one hand, this is just $\sum_{i=1}^{(p-1)/2} i \bmod 2$.
- On the other hand, the sum equals

$$\begin{aligned} & mp + \sum_{i=1}^{(p-1)/2} \left(iq - p \left\lfloor \frac{iq}{p} \right\rfloor \right) \\ \equiv \; & mp + \left(q \sum_{i=1}^{(p-1)/2} i - p \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) \bmod 2. \end{aligned}$$

  - m of the $iq \bmod p$ are replaced by $p - iq \bmod p$.
  - But signs are irrelevant under mod2.
  - m is as in Lemma 70 (p. 581).

The Proof (continued)

- Ignore odd multipliers to make the sum equal

$$m + \left(\sum_{i=1}^{(p-1)/2} i - \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) \bmod 2.$$

- Equate the above with $\sum_{i=1}^{(p-1)/2} i$ modulo 2.
- Now simplify to obtain

$$m \equiv \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \bmod 2.$$

The Proof (continued)

- $\sum_{i=1}^{(p-1)/2} \lfloor \frac{iq}{p} \rfloor$ is the number of integral points *below* the line

$$y = (q/p)\,x$$

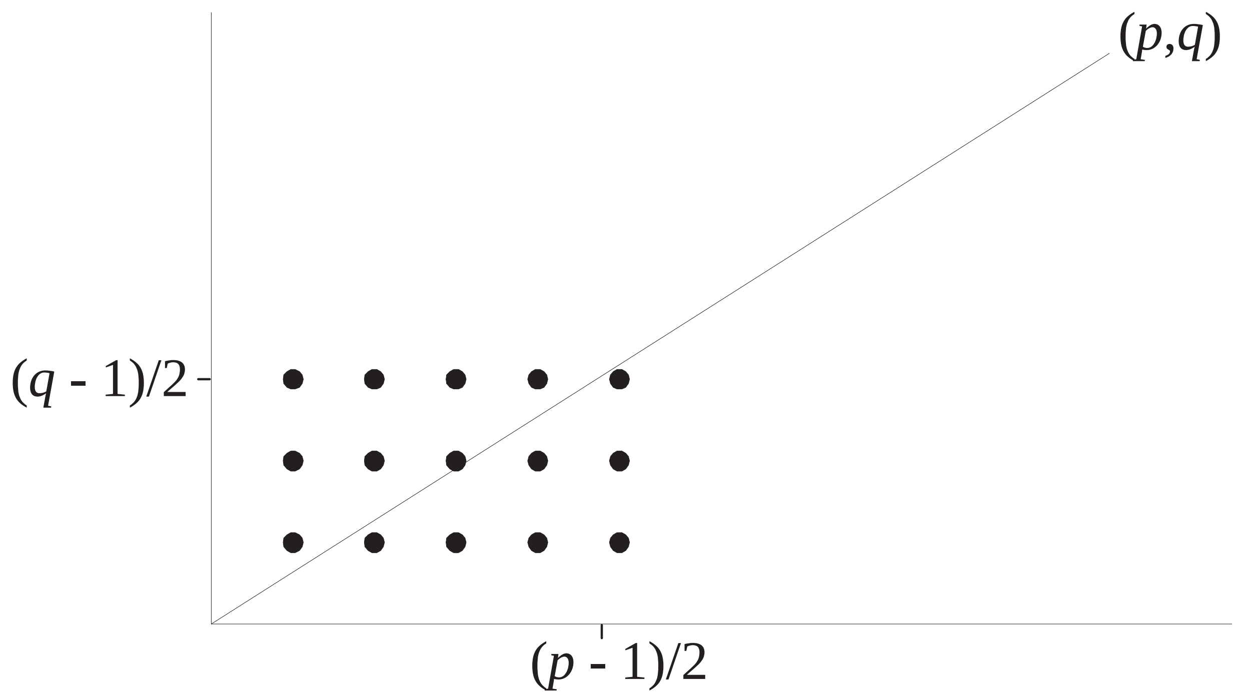
for $1 \leq x \leq (p-1)/2$.

- Gauss's lemma (p. 581) says $(q \,|\, p) = (-1)^m$.
- Repeat the proof with p and q reversed.
- Then $(p \,|\, q) = (-1)^{m'}$, where m' is the number of integral points *above* the line $y = (q/p)\,x$ for $1 \leq y \leq (q-1)/2$.

The Proof (concluded)

- As a result,
$$(p \,|\, q)(q \,|\, p) = (-1)^{m+m'}.$$
- But $m + m'$ is the total number of integral points in the $[1, \frac{p-1}{2}] \times [1, \frac{q-1}{2}]$ rectangle, which is
$$\frac{p-1}{2} \, \frac{q-1}{2}.$$

Eisenstein's Rectangle



Above, $p = 11$, $q = 7$, $m = 7$, $m' = 8$.