Theory of Computation

Midterm Examination on October 14, 2022
Fall Semester, 2022

Problem 1 (20 points) Prove that the following language L is undecidable:

$$L = \{\, M; x; n \,:\, \text{a TM } M \text{ with input } x \text{ executes the } n\text{th instruction of } M \,\}.$$

Use reduction from the halting problem $H = \{\, M; x : M(x) \neq \nearrow \,\}$.

Proof: Suppose that L is decidable. We now reduce the halting problem to L. Consider an instance $M; x$. Then replace all instructions $\delta(q, s) = (r, t, D)$, where r is a "yes," a "no," or an h, with $\delta(q, s) = (Q, t, D)$, where Q is a new state. Then add instructions which make the head move to the beginning of the tape (with symbol $\triangleright$) while remaining at state Q. Let k be the number of instructions of the aforesaid machine. Finally, add the instruction $\delta(Q, \triangleright) = (h, \triangleright, -)$ numbered $n = k + 1$. Call this modified machine M'. Now we construct a TM M'' such that

$$M''(M'; x; n) = \begin{cases} \text{"yes", if } M'(x) \text{ executes the } n\text{th instruction;} \\ \text{"no", otherwise.} \end{cases}$$

Clearly $M'; x; n \in L$ if and only if $M; x \in H$, a contradiction. Hence L is undecidable. ■

Problem 2 (20 points) Answer the following questions.

(1) Write down the property of being a polynomially decidable language.

(2) Use Rice's theorem to prove that this property is undecidable.

Proof:

(1) The property of being a polynomially decidable language is

$$\{\, L : \text{a TM } M \text{ decides } L \text{ in polynomial time} \,\}.$$

(2) We know $P \subsetneq E$ of the lecture notes, which also implies there are problems in E but not in P. Hence the said property is nontrivial and Rice's theorem applies.

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Problem 3 (20 points) We call a boolean function $f : \{0,1\}^k \to \{0,1\}$ symmetric if $f(x_1, x_2, \ldots, x_k)$ depends only on $\sum_{i=1}^{k} = x_i$. How many symmetric boolean functions of k variables are there?

Proof: 2^{k+1}. ■

Problem 4 (20 points) Suppose L_1 is undecidable and $L_1 \subseteq L_2$. Is L_2 undecidable? Either prove it or give examples with proofs.

Proof: It depends. When $L_2 = \Sigma^*$, L_2 is decidable: Just answer "yes." If $L_2 - L_1$ is decidable, then L_2 is undecidable. This is because:

- Clearly,

$$x \in L_1 \text{ if and only if } x \in L_2 \text{ and } x \notin L_2 - L_1.$$

- Therefore, if L_2 were decidable, then L_1 would be.

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Problem 5 (20 points) Let ϕ_1 and ϕ_2 be arbitrary boolean expressions. We say ϕ_1 and ϕ_2 are equivalent, written in $\phi_1 \equiv \phi_2$, if for any truth assignment T appropriate to both of them, $T \models \phi_1$ if and only if $T \models \phi_2$. Prove that $\phi_1 \equiv \phi_2$ if and only if $\phi_1 \Leftrightarrow \phi_2$ is tautology. (Recall that $\phi_1 \Leftrightarrow \phi_2$ is a short hand for $(\phi_1 \Rightarrow \phi_2) \wedge (\phi_2 \Rightarrow \phi_1)$.)

Proof:

- (If) Suppose that $\phi_1 \Leftrightarrow \phi_2$ is tautology. Assume that $T \models \phi_1$. Then T satisfies ϕ_2 because $\phi_1 \Leftrightarrow \phi_2$ is a tautology. Similarly, T satisfies ϕ_1 if T satisfies ϕ_2. Thus $\phi_1 \equiv \phi_2$.

- (Only if) Suppose $\phi_1 \equiv \phi_2$. For any truth assignment T such that $T \models \phi_1$ and $T \models \phi_2$, T surely satisfies $\phi_1 \wedge \phi_2$. On the other hand, for any truth assignment T such that $T \not\models \phi_1$ and $T \not\models \phi_2$, T satisfies $\neg\phi_1 \wedge \neg\phi_2$. There are no other possibilities because $\phi_1 \equiv \phi_2$. So all truth assignments satisfy $(\phi_1 \wedge \phi_2) \vee (\neg\phi_1 \wedge \neg\phi_2)$ which is logically equivalent to $\phi_1 \Leftrightarrow \phi_2$. Hence $\phi_1 \Leftrightarrow \phi_2$ is tautology.

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