Theory of Computation

Midterm Examination on October 28, 2021
Fall Semester, 2021

Problem 1 (20 points) Prove that $H^\varepsilon = \{ M : M \text{ halts on } \varepsilon \}$ is undecidable by reduction from the halting set $H = \{ M; x : M(x) \neq \nearrow \}$. (Recall that ε is the empty string.)

Proof: Given the question "$M; x \in H$?" we construct the following machine:

$$M_x(y) : M(x).$$

M halts on x if and only if M_x halts on ε. In other words, $M; x \in H$ if and only if $M_x \in H^\varepsilon$. So if H^ε were recursive, H would be recursive, a contradiction. ■

Problem 2 (20 points) Answer the following questions.

(1) Write down the property of being a recursive language.

(2) Use Rice's theorem to prove that this property is undecidable.

Proof:

(1) The property of being a *recursive* languages is

$$\{ L : L = L(M) \text{ for some TM } M \text{ and } L \text{ is recursive} \}.$$

(2) Because $R \subseteq RE$, the above property is a subset of RE. We also know that the halting problem is recursively enumerable but not recursive. Finally, we also know that recursive languages exist (such as primality, etc.). Hence the said property is nontrivial. So Rice's theorem applies.

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Problem 3 (20 points) Prove that if L_1 and L_2 are recursively enumerable languages, then so is $L_1 \cup L_2$.

Proof: Assume that TM M_1 accepts L_1 and TM M_2 accepts L_2. We then construct another TM M' which simulates M_1 and M_2 in an interleaving style:

1. if $x \in L_1$, then $M'(x) = M_1(x) =$ "yes";
2. if $x \in L_2$, then $M'(x) = M_2(x) =$ "yes";
3. otherwise, $M'(x) = \nearrow$.

So M' accepts $L_1 \cup L_2$. The claim is proved. ■

Problem 4 (20 points) Prove that $L = \{ M; x \mid M(x) = \text{"yes"} \}$ is undecidable. (Do not use Rice's theorem nor reduction from another undecidable problem. Instead, prove it as the undecidability of the halting problem was proved in the slides.)

Proof: Suppose L is recursive. Then there is a TM M_L that decides L. Consider the following program $D(M)$ which calls M_L:

if $M_L(M; M) =$ "yes" **then**
    $D(M) =$ "no";
else
    $D(M) =$ "yes";
end if

Consider $D(D)$: $D(D) = \text{"no"} \Rightarrow M_L(D; D) = \text{"yes"} \Rightarrow D; D \in L \Rightarrow D(D) = \text{"yes"}$, a contradiction. $D(D) = \text{"yes"} \Rightarrow M_L(D; D) = \text{"no"} \Rightarrow D; D \notin L \Rightarrow D(D) = \text{"no"}$, another contradiction. Hence, L is undecidable. ■

Problem 5 (20 points) Answer the following questions.

(1) (5 points) Consider the following boolean functions

$$f : \{ \texttt{true}, \texttt{false} \}^n \to \{ \texttt{true}, \texttt{false} \}^{\sqrt{n}}.$$

How many such functions are there? For simplicity, assume that $\sqrt{x}$ is an integer. Same below. (Use parentheses to avoid ambiguity.)

(2) (15 points) What is a good lower bound for any circuit that computes such functions (assuming $n \geq 2$ or any convenient constant)?

Proof:

(1) $2^{\left(2^n\sqrt{n}\right)}$.

(2) From slide p. 213, it suffices to show that $((n+5)\times m^2)^m < 2^{(2^n\sqrt{n})}$ when $m = 2^{n-1}/\sqrt{n+5}$:

$$\begin{aligned} m\log_2((n+5)\times m^2) &= \left(\frac{n-1}{\sqrt{n+5}}\right)\times 2^n \\ &< \sqrt{n}\times 2^n \end{aligned}$$

for $n \geq 2$. So a lower bound is $2^{n-1}/\sqrt{n+5}$.

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