Relations between Complexity Classes

It is, I own, not uncommon to be
wrong in theory
and right in practice.
— Edmund Burke (1729–1797),
A Philosophical Enquiry into the Origin of Our Ideas of the Sublime and Beautiful (1757)

The problem with QE is
it works in practice,
but it doesn't work in theory.
— Ben Bernanke (2014)

Proper (Complexity) Functions

- We say that $f : \mathbb{N} \to \mathbb{N}$ is a **proper (complexity) function** if the following hold:
  - f is nondecreasing.
  - There is a k-string TM M_f such that $M_f(x) = \sqcap^{f(|x|)}$ for any x.[a]
  - M_f halts after $O(|x| + f(|x|))$ steps.
  - M_f uses $O(f(|x|))$ space besides its input x.
- M_f's behavior depends only on $|x|$ not x's contents.
- M_f's running time is bounded by $f(n)$.

[a]The textbook calls "$\sqcap$" the quasi-blank symbol. The use of $M_f(x)$ will become clear in Proposition 17 (p. 231).

Examples of Proper Functions

- Most "reasonable" functions are proper: c, $\lceil \log n \rceil$, polynomials of n, 2^n, $\sqrt{n}$, $n!$, etc.
- If f and g are proper, then so are $f + g$, fg, and 2^g.[a]
- Nonproper functions when serving as the time bounds for complexity classes spoil "theory building."
  - For example, $\text{TIME}(f(n)) = \text{TIME}(2^{f(n)})$ for some recursive function f (the **gap theorem**).[b]
- Only proper functions f will be used in $\text{TIME}(f(n))$, $\text{SPACE}(f(n))$, $\text{NTIME}(f(n))$, and $\text{NSPACE}(f(n))$.

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[a] For $f(g(n))$, we need to add $f(n) \geq n$.

[b] Trakhtenbrot (1964); Borodin (1972). Theorem 7.3 on p. 145 of the textbook proves it.

Precise Turing Machines

- A TM M is **precise** if there are functions f and g such that for every $n \in \mathbb{N}$, for every x of length n, and for every computation path of M,
  - M halts after *precisely* $f(n)$ steps,[a] and
  - All of its strings are of length precisely $g(n)$ at halting.[b]
    * Recall that if M is a TM with input and output, we exclude the first and last strings.
- M can be deterministic or nondeterministic.

[a]Fully time constructible (Hopcroft & Ullman, 1979).
[b]Fully space constructible (Hopcroft & Ullman, 1979).

Precise TMs Are General

Proposition 17 *Suppose a TM*[a] M *decides* L *within time (space)* $f(n)$, *where* f *is proper. Then there is a precise TM* M' *which decides* L *in time* $O(n + f(n))$ *(space* $O(f(n))$, *respectively).*

- M' on input x first simulates the TM M_f associated with the proper function f on x.
- M_f's output, of length $f(|\,x\,|)$, will serve as a "yardstick" or an "alarm clock."

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[a]Deterministic or nondeterministic.

The Proof (continued)

- Then M' simulates $M(x)$.
- $M'(x)$ halts when and only when the alarm clock runs out—even if M halts earlier.
- If f is a time bound:
  - The simulation of each step of M on x is matched by advancing the cursor on the "clock" string.
  - Because M' stops at the moment the "clock" string is exhausted—even if $M(x)$ stops earlier, it is precise.
  - The time bound is therefore $O(|x| + f(|x|))$.

The Proof (concluded)

- If f is a space bound (sketch):
  - M' simulates M *on* the quasi-blanks of M_f's output string.[a]
  - The total space, not counting the input string, is $O(f(n))$.
  - But we still need a way to make sure there is no infinite loop even if M does not halt.[b]

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[a]This is to make sure the space bound is precise.

[b]See the proof of Theorem 24 (p. 250).

Important Complexity Classes

- We write expressions like n^k to denote the union of all complexity classes, one for each value of k.
- For example,

$$\mathrm{NTIME}(n^k) \stackrel{\Delta}{=} \bigcup_{j>0} \mathrm{NTIME}(n^j).$$

Important Complexity Classes (concluded)

$$
\begin{aligned}
\text{P} &\stackrel{\Delta}{=} \text{TIME}(n^k), \\
\text{NP} &\stackrel{\Delta}{=} \text{NTIME}(n^k), \\
\text{PSPACE} &\stackrel{\Delta}{=} \text{SPACE}(n^k), \\
\text{NPSPACE} &\stackrel{\Delta}{=} \text{NSPACE}(n^k), \\
\text{E} &\stackrel{\Delta}{=} \text{TIME}(2^{kn}), \\
\text{EXP} &\stackrel{\Delta}{=} \text{TIME}(2^{n^k}), \\
\text{NEXP} &\stackrel{\Delta}{=} \text{NTIME}(2^{n^k}), \\
\text{L} &\stackrel{\Delta}{=} \text{SPACE}(\log n), \\
\text{NL} &\stackrel{\Delta}{=} \text{NSPACE}(\log n).
\end{aligned}
$$

Complements of Nondeterministic Classes

- Recall that the complement of L, or $\bar{L}$, is the language $\Sigma^* - L$.
  - SAT COMPLEMENT is the set of unsatisfiable boolean expressions.
- R, RE, and coRE are distinct.[a]
  - Again, coRE contains the complements of *languages* in RE, *not* languages that are not in RE.

[a] Recall p. 166.

The Co-Classes

- For any *complexity* class $\mathcal{C}$, co$\mathcal{C}$ denotes the class

$$\{ L : \bar{L} \in \mathcal{C} \}.$$

- Clearly, if $\mathcal{C}$ is a *deterministic* time or space *complexity class*, then $\mathcal{C} = \text{co}\mathcal{C}$.
  - They are said to be **closed under complement**.
  - A deterministic TM deciding L can be converted to one that decides $\bar{L}$ within the same time or space bound by reversing the "yes" and "no" states.[a]
- Whether *non*deterministic classes for time are closed under complement is not known.

[a]Recall p. 163.

Comments

- As
$$\text{co}\mathcal{C} = \{\, L : \bar{L} \in \mathcal{C} \,\},$$
$L \in \mathcal{C}$ if and only if $\bar{L} \in \text{co}\mathcal{C}$.
- But it is *not* true that $L \in \mathcal{C}$ if and only if $L \notin \text{co}\mathcal{C}$.
  - co$\mathcal{C}$ is not defined as $\bar{\mathcal{C}}$.
- For example, suppose $\mathcal{C} = \{\{\, 2, 4, 6, 8, 10, \ldots \,\}, \ldots\}$.
- Then co$\mathcal{C} = \{\{\, 1, 3, 5, 7, 9, \ldots \,\}, \ldots\}$.
- But $\bar{\mathcal{C}} = 2^{\{\, 1,2,3,\ldots \,\}} - \{\{\, 2, 4, 6, 8, 10, \ldots \,\}, \ldots\}$.

The Quantified Halting Problem

- Let $f(n) \geq n$ be proper.
- Define

$$H_f \triangleq \{ M; x : M \text{ accepts input } x \text{ after at most } f(|x|) \text{ steps} \},$$

  where M is deterministic.
- Assume the input is binary as usual.

$H_f \in \mathsf{TIME}(f^3(n))$

- For each input $M; x$, we simulate M on x with an alarm clock of length $f(|\,x\,|)$.
  - Use the single-string simulator (p. 87), the universal TM (p. 142), and the linear speedup theorem (p. 97).
  - Our simulator accepts $M; x$ if and only if M accepts x before the alarm clock runs out.
- From p. 94, the total running time is $O(\ell_M k_M^2 f^2(n))$, where ℓ_M is the length to encode each symbol or state of M and k_M is M's number of strings.
- As $\ell_M k_M^2 = O(n)$, the running time is $O(f^3(n))$, where the constant is independent of M.

$$H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$$

- Suppose TM M_{H_f} decides H_f in time $f(\lfloor n/2 \rfloor)$.
- Consider machine:

$$
\begin{array}{ll}
D_f(M) & \{ \\
 & \quad \textbf{if } M_{H_f}(M; M) = \text{"yes"} \\
 & \quad \textbf{then } \text{"no"}; \\
 & \quad \textbf{else } \text{"yes"}; \\
\} &
\end{array}
$$

The Proof (continued)

- $M_{H_f}(M; M)$ runs in time $f(\lfloor \frac{2n+1}{2} \rfloor) = f(n)$, where $n = |\, M \,|$.[a]
- By construction, $D_f(M)$ runs in the same amount of time as $M_{H_f}(M; M)$, i.e., $f(n)$, where $n = |\, M \,|$.

[a]Mr. Hsiao-Fei Liu (`F92922019`) and Mr. Hong-Lung Wang (`F92922085`) pointed out on October 6, 2004, that this estimation (and the text's Lemma 7.2) forgets to include the time to write down $M; M$.

The Proof (concluded)

- First, suppose $D_f(D_f) =$ "yes".
- This implies

$$D_f; D_f \notin H_f.$$

- Thus D_f does not accept D_f within time $f(\mid D_f \mid)$.
- But $D_f(D_f)$ stops in time $f(\mid D_f \mid)$ with an answer.
- Hence $D_f(D_f) =$ "no", a contradiction
- Similarly, $D_f(D_f) =$ "no" $\Rightarrow D_f(D_f) =$ "yes."

The Time Hierarchy Theorem

Theorem 18 *If $f(n) \geq n$ is proper, then*

$$\text{TIME}(f(n)) \subsetneq \text{TIME}(f^3(2n+1)).$$

- The quantified halting problem makes it so.

Corollary 19 $\text{P} \subsetneq \text{E}$.

- $\text{P} \subseteq \text{TIME}(2^n)$ because $\text{poly}(n) \leq 2^n$ for n large enough.
- But by Theorem 18,

$$\text{TIME}\left(2^n\right) \subsetneq \text{TIME}\left((2^{2n+1})^3\right) \subseteq \text{E}.$$

- So $\text{P} \subsetneq \text{E}$.

The Space Hierarchy Theorem

Theorem 20 (Hennie & Stearns, 1966) *If $f(n)$ is proper, then*

$$\text{SPACE}(f(n)) \subsetneq \text{SPACE}(f(n) \log f(n)).$$

Corollary 21 $\text{L} \subsetneq \text{PSPACE}$.

Nondeterministic Time Hierarchy Theorems

Theorem 22 (Cook, 1973) $\text{NTIME}(n^r) \subsetneq \text{NTIME}(n^s)$ *whenever* $1 \leq r < s$.

Theorem 23 (Seiferas, Fischer, & Meyer, 1978) *If* $T_1(n)$ *and* $T_2(n)$ *are proper, then*

$$\text{NTIME}(T_1(n)) \subsetneq \text{NTIME}(T_2(n))$$

whenever $T_1(n+1) = o(T_2(n))$.

The Reachability Method

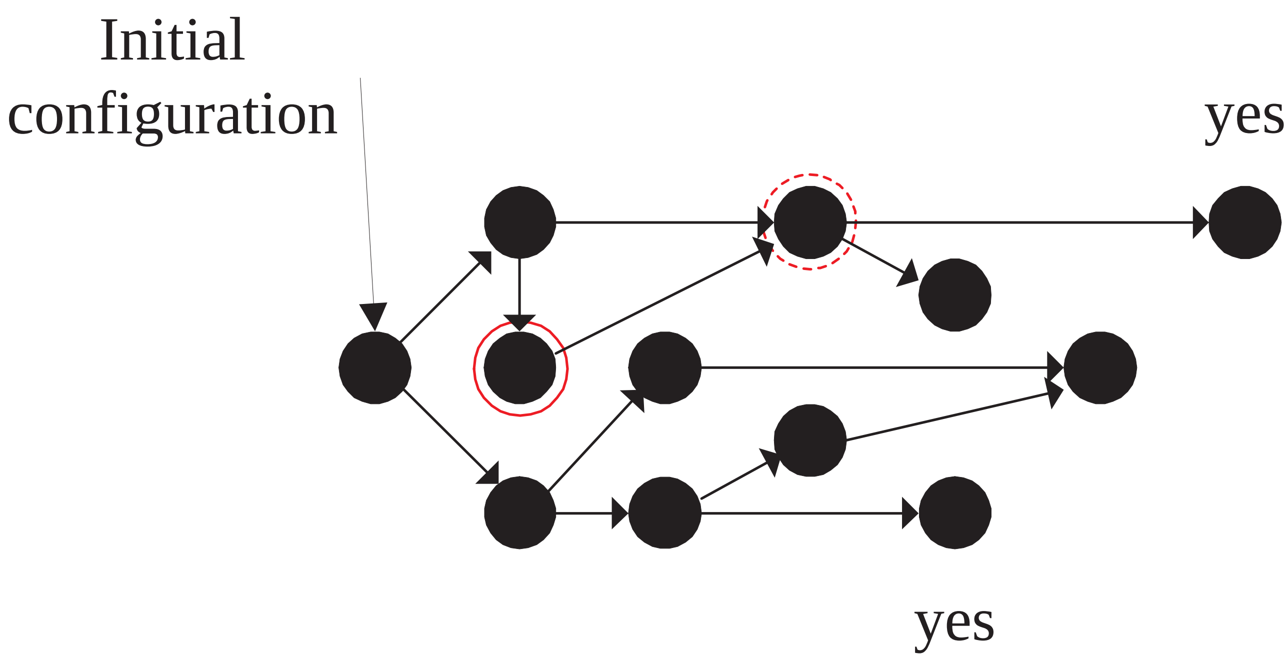
- The computation of a time-bounded TM can be represented by a directed graph.
- The TM's configurations constitute the nodes.
- There is a directed edge from node x to node y if x yields y in one step.
- The start node representing the initial configuration has zero in-degree.

The Reachability Method (concluded)

- When the TM is nondeterministic, a node may have an out-degree greater than one.
  - The graph is the same as the computation tree earlier.
  - But identical configurations are merged into one node.[a]
- So M accepts the input if and only if there is a path from the start node to a node with a "yes" state.
- It is the reachability problem.

[a]So we end up with a graph not a tree.

Illustration of the Reachability Method



Relations between Complexity Classes

Theorem 24 *Suppose $f(n)$ is proper. Then*

1. $\text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n))$, $\text{TIME}(f(n)) \subseteq \text{NTIME}(f(n))$.
2. $\text{NTIME}(f(n)) \subseteq \text{SPACE}(f(n))$.
3. $\text{NSPACE}(f(n)) \subseteq \text{TIME}(k^{\log n + f(n)})$.

- Proof of 2:
  - Explore the computation *tree* of the NTM for "yes."
  - Specifically, generate an $f(n)$-bit sequence denoting the nondeterministic choices over $f(n)$ steps.

Proof of Theorem 24(2)

- (continued)
  - Simulate the NTM based on the choices.
  - Recycle the space and repeat the above steps.
  - Halt with "yes" when a "yes" is encountered or "no" if the tree is exhausted.
  - Each path simulation consumes at most $O(f(n))$ space because it takes $O(f(n))$ time.
  - The total space is $O(f(n))$ because space is recycled.

Proof of Theorem 24(3)

- Let k-string NTM

$$M = (K, \Sigma, \Delta, s)$$

with input and output decide $L \in \text{NSPACE}(f(n))$.

- Use the reachability method on the configuration graph of M on input x of length n.

- A configuration is a $(2k+1)$-tuple

$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$

Proof of Theorem 24(3) (continued)

- We only care about

$$(q, i, w_2, u_2, \ldots, w_{k-1}, u_{k-1}),$$

where i is an integer between 0 and n for the position of the *first* cursor.

- The number of configurations is therefore at most

$$|\,K\,| \times (n+1) \times |\,\Sigma\,|^{2(k-2)f(n)} = O(c_1^{\log n + f(n)}) \tag{2}$$

for some $c_1 > 1$, which depends on M.

- Add edges to the configuration graph based on M's transition function.

Proof of Theorem 24(3) (concluded)

- $x \in L \Leftrightarrow$ there is a path in the configuration graph from the initial configuration to a configuration of the form ("yes", $i, \ldots$).[a]
- This is REACHABILITY on a graph with $O(c_1^{\log n + f(n)})$ nodes.
- It is in TIME$(c^{\log n + f(n)})$ for some $c > 1$ because REACHABILITY $\in$ TIME(n^j) for some j and

$$\left[c_1^{\log n + f(n)} \right]^j = (c_1^j)^{\log n + f(n)}.$$

[a]There may be many of them.

Space-Bounded Computation and Proper Functions

- In the definition of *space-bounded* computations earlier (p. 116), the TMs are not required to halt at all.
- When the space is bounded by a proper function f, computations can be assumed to halt:
  - Run the TM associated with f to produce a quasi-blank output of length $f(n)$ first.
  - The space-bounded computation must repeat a configuration if it runs for more than $c^{\log n+f(n)}$ steps for some $c > 1$.[a]

[a]See Eq. (2) on p. 253.

Space-Bounded Computation and Proper Functions (concluded)

- (continued)
  - So an infinite loop occurs during simulation for a computation path longer than $c^{\log n + f(n)}$ steps.
  - Hence we only simulate up to $c^{\log n + f(n)}$ time steps per computation path.

A Grand Chain of Inclusions[a]

- It is an easy application of Theorem 24 (p. 250) that

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXP.$$

- By Corollary 21 (p. 245), we know $L \subsetneq PSPACE$.
- So the chain must break somewhere between L and EXP.
- It is suspected that all four inclusions are proper.
- But there are no proofs yet.

[a]With input from Mr. Chin-Luei Chang (B89902053, R93922004, D95922007) on October 22, 2004.

What Is Wrong with the Proof?[a]

- By Theorem 24(2) (p. 250),

$$\mathrm{NL} \subseteq \mathrm{TIME}\left(k^{O(\log n)}\right) \subseteq \mathrm{TIME}\,(n^{c_1})$$

for some $c_1 > 0$.

- By Theorem 18 (p. 244),

$$\mathrm{TIME}\,(n^{c_1}) \subsetneq \mathrm{TIME}\,(n^{c_2}) \subseteq \mathrm{P}$$

for some $c_2 > c_1$.

- So

$$\mathrm{NL} \neq \mathrm{P}.$$

[a]Contributed by Mr. Yuan-Fu Shao (R02922083) on November 11, 2014.

What Is Wrong with the Proof? (concluded)

- Recall from p. 234 that $\mathrm{TIME}(k^{O(\log n)})$ is a shorthand for

$$\bigcup_{j>0} \mathrm{TIME}\left(j^{O(\log n)}\right).$$

- So the correct proof runs more like

$$\mathrm{NL} \subseteq \bigcup_{j>0} \mathrm{TIME}\left(j^{O(\log n)}\right) \subseteq \bigcup_{c>0} \mathrm{TIME}\left(n^{c}\right) = \mathrm{P}.$$

- And

$$\mathrm{NL} \neq \mathrm{P}$$

no longer follows.

Nondeterministic and Deterministic Space

- By Theorem 6 (p. 132),
$$\mathrm{NTIME}(f(n)) \subseteq \mathrm{TIME}(c^{f(n)}),$$
an exponential gap.
- There is no proof yet that the exponential gap is inherent.
- How about NSPACE vs. SPACE?
- Surprisingly, the relation is only quadratic—a polynomial—by Savitch's theorem.

Savitch's Theorem

Theorem 25 (Savitch, 1970)

$$\text{REACHABILITY} \in \text{SPACE}(\log^2 n).$$

- Let $G(V, E)$ be a graph with n nodes.
- For $i \geq 0$, let
$$\text{PATH}(x, y, i)$$
mean there is a path from node x to node y of length at most 2^i.
- There is a path from x to y if and only if
$$\text{PATH}(x, y, \lceil \log n \rceil)$$
holds.

The Proof (continued)

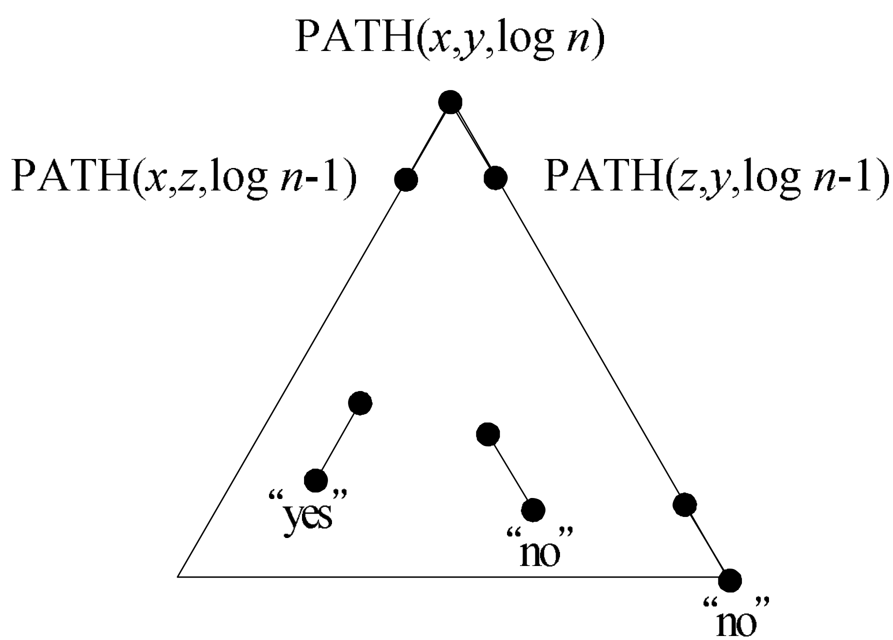
- For $i > 0$, PATH(x, y, i) if and only if there exists a z such that PATH$(x, z, i-1)$ and PATH$(z, y, i-1)$.
- For PATH$(x, y, 0)$, check the input graph or if $x = y$.
- Compute PATH$(x, y, \lceil \log n \rceil)$ with a depth-first search on a graph with nodes (x, y, i)s (see next page).[a]
- Like stacks in recursive calls, we keep only the current path's (x, y, i)s.

[a]Contributed by Mr. Chuan-Yao Tan on October 11, 2011.

The Proof (continued): Algorithm for PATH(x, y, i)

```
if i = 0 then
    if x = y or (x, y) ∈ E then
        return true;
    else
        return false;
    end if
else
    for z = 1, 2, ..., n do
        if PATH(x, z, i − 1) and PATH(z, y, i − 1) then
            return true;
        end if
    end for
    return false;
end if
```

The Proof (continued)



The Proof (concluded)

- The space requirement is proportional to the depth of the tree ($\lceil \log n \rceil$) times the size of the items stored at each node.
- Depth is $\lceil \log n \rceil$, and each node (x, y, i) needs space $O(\log n)$.
- The total space is $O(\log^2 n)$.

The Relation between Nondeterministic and Deterministic Space Is Only Quadratic

Corollary 26 *Let $f(n) \geq \log n$ be proper. Then*

$$\mathrm{NSPACE}(f(n)) \subseteq \mathrm{SPACE}(f^2(n)).$$

- Apply Savitch's proof to the configuration graph of the NTM on its input.
- From p. 253, the configuration graph has $O(c^{f(n)})$ nodes; hence each node takes space $O(f(n))$.
- But if we construct *explicitly* the whole graph before applying Savitch's theorem, we get $O(c^{f(n)})$ space!

The Proof (continued)

- The way out is *not* to generate the graph at all.
- Instead, keep the graph implicit.
- We checked node connectedness only when $i = 0$ on p. 263, by examining the input graph G.
- Suppose we are given configurations x and y.
- Then we go over the Turing machine's program to determine if there is an instruction that can turn x into y in one step.[a]
- So connectivity is checked locally and on demand.

[a]Thanks to a lively class discussion on October 15, 2003.

The Proof (continued)

- The z variable in the algorithm on p. 263 simply runs through all possible valid configurations.
  - Let $z = 0, 1, \ldots, O(c^{f(n)})$.
  - Make sure z is a valid configuration before proceeding with it.[a]
    - Adopt the same width for each symbol and state of the NTM and for the cursor position on the input string.[b]
  - If it is not, advance to the next z.

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[a] Thanks to a lively class discussion on October 13, 2004.

[b] Contributed by Mr. Jia-Ming Zheng (`R04922024`) on October 17, 2017.

The Proof (concluded)

- Each z has length $O(f(n))$.
- So each node needs space $O(f(n))$.
- The depth of the recursive call on p. 263 is $O(\log c^{f(n)})$, which is $O(f(n))$.
- The total space is therefore $O(f^2(n))$.

Implications of Savitch's Theorem

Corollary 27 *PSPACE = NPSPACE.*

- Nondeterminism is less powerful with respect to space.
- Nondeterminism may be very powerful with respect to time as it is not known if P = NP.

Nondeterministic Space Is Closed under Complement

- Closure under complement is trivially true for deterministic complexity classes.[a]
- It is known that[b]

$$\text{coNSPACE}(f(n)) = \text{NSPACE}(f(n)). \tag{3}$$

- So

$$\text{coNL} = \text{NL}.$$

- But it is not known whether coNP = NP.[c]

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[a] Recall p. 237.

[b] Szelepscényi (1987); Immerman (1988).

[c] If P = NP, then coNP = NP. Contributed by Mr. Yu-Ming Lu (R06723032, D08922008) on October 21, 2021.