# Complementing a TM's Halting States

- Let $M$ decide $L$, and $M'$ be $M$ after "yes" $\leftrightarrow$ "no".
- If $M$ is *deterministic*, then $M'$ decides $\bar{L}$.[a]
  - So $M$ and $M'$ decide languages that complement each other.
- But if $M$ is an NTM, then $M'$ may not decide $\bar{L}$.
  - It is possible that $M$ and $M'$ accept the same input $x$ (see next page).
  - So $M$ and $M'$ may accept languages that are *not* even disjoint.

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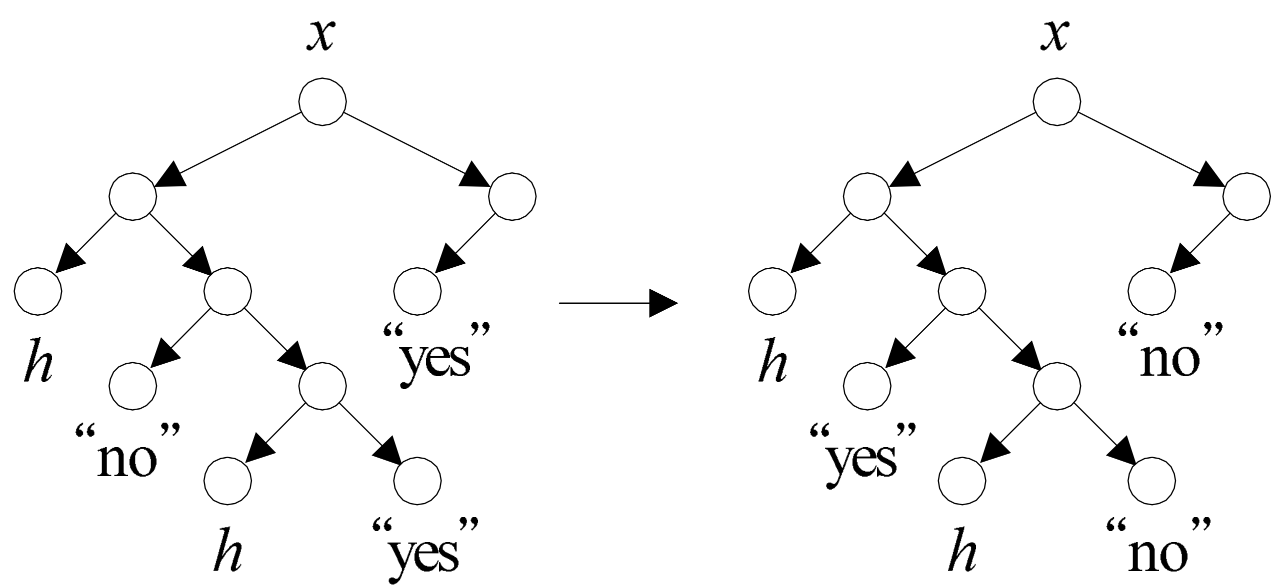
[a]By the definition on p. 53, $M$ must halt on all inputs.



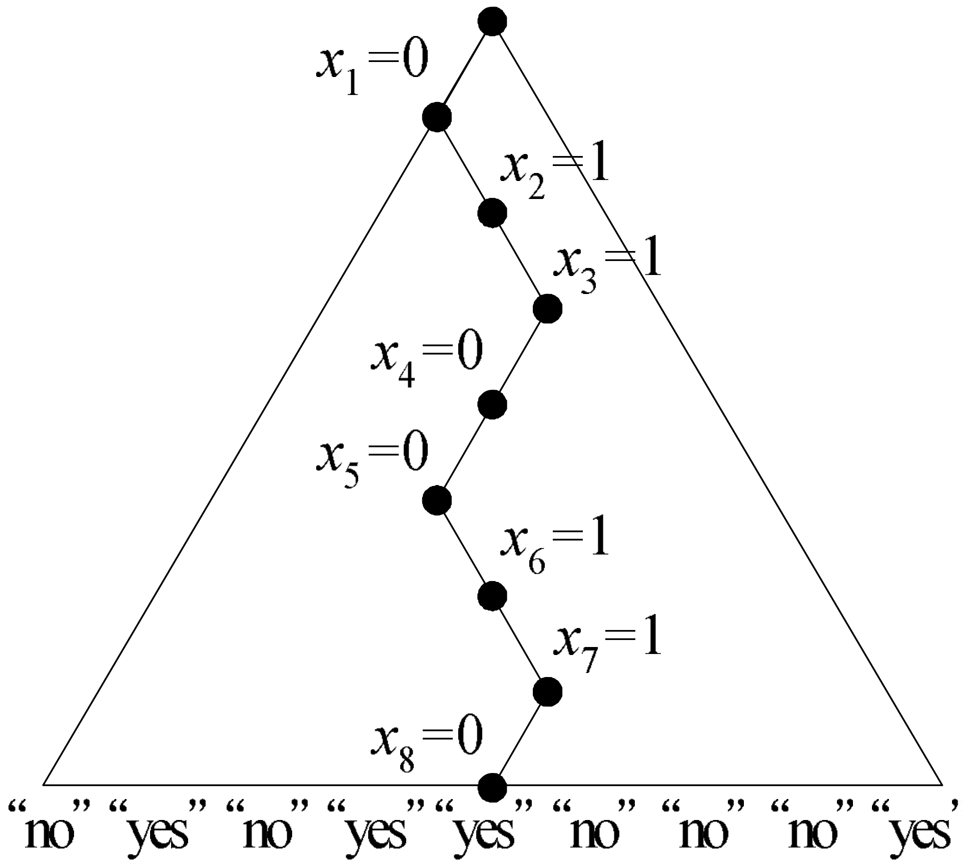
x
h
"no"
h
"yes"
"yes"
x
h
"yes"
h
"no"
"no"

# A Nondeterministic Algorithm for Satisfiability

$\phi$ is a boolean formula with $n$ variables.

**for** $i = 1, 2, \ldots, n$ **do**
    Guess $x_i \in \{ 0, 1 \}$; {Nondeterministic choices.}
**end for**
{Verification:}
**if** $\phi(x_1, x_2, \ldots, x_n) = 1$ **then**
    "yes";
**else**
    "no";
**end if**

## Computation Tree for Satisfiability



## Analysis

- Recall that $\phi$ is satisfiable if and only if there is a truth assignment that satisfies $\phi$.
- The computation tree is a complete binary tree of depth $n$.
- Every computation path corresponds to a particular truth assignment[a] out of $2^n$.

[a]Equivalently, a sequence of nondeterministic choices.

## Analysis (concluded)

- The algorithm decides language

$$\{\, \phi : \phi \text{ is satisfiable} \,\}.$$

  - Suppose $\phi$ is satisfiable.
    * There is a truth assignment that satisfies $\phi$.
    * So there is a computation path that results in "yes."
  - Suppose $\phi$ is not satisfiable.
    * That means every truth assignment makes $\phi$ false.
    * So every computation path results in "no."
- General paradigm: Guess a "proof" then verify it.

## The Traveling Salesman Problem

- We are given $n$ cities $1, 2, \ldots, n$ and integer distance $d_{ij}$ between any two cities $i$ and $j$.
- Assume $d_{ij} = d_{ji}$ for convenience.
- The **traveling salesman problem** (TSP) asks for the total distance of the shortest tour of the cities.[a]
- The decision version TSP (D) asks if there is a tour with a total distance at most $B$, where $B$ is an input.[b]

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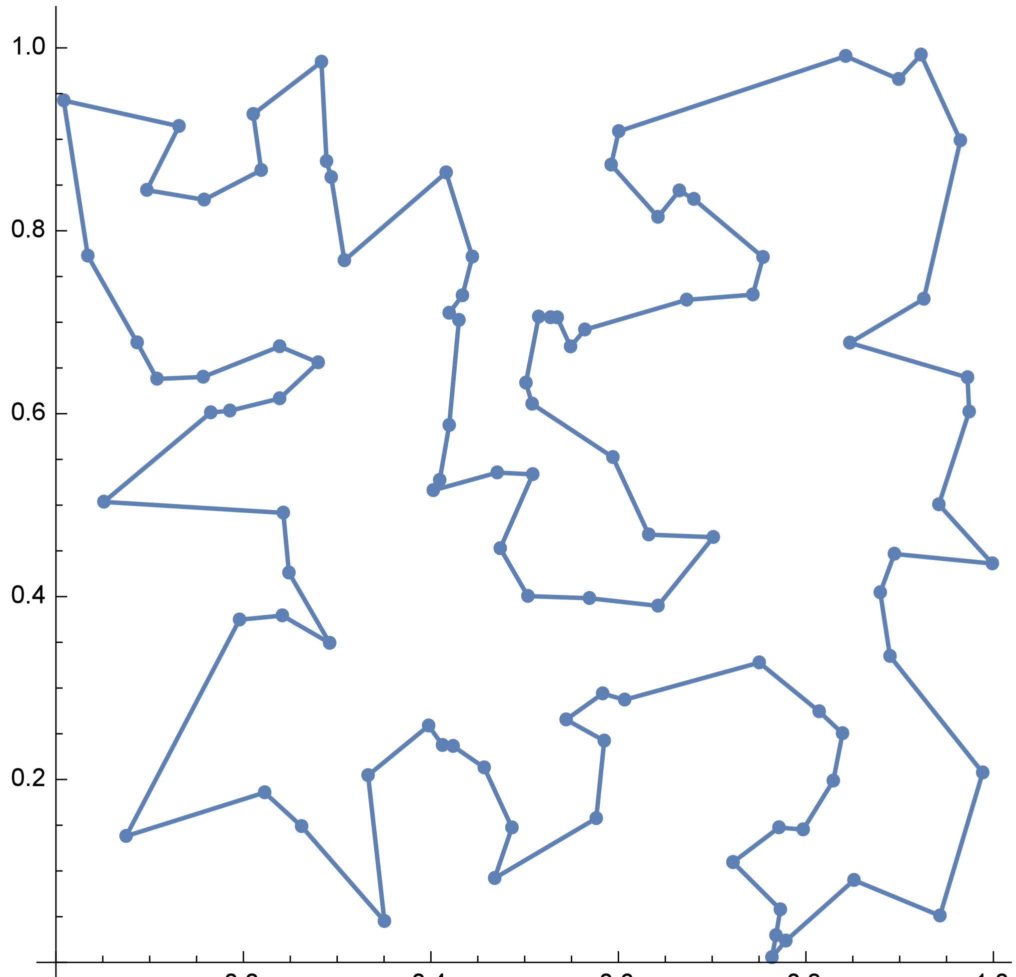
[a] Each city is visited exactly once.

[b] Both problems are extremely important. They are equally hard (p. 415 and p. 516).

# A Shortest Tour



## A Nondeterministic Algorithm for TSP (D)

**for** $i = 1, 2, \ldots, n$ **do**
  Guess $x_i \in \{ 1, 2, \ldots, n \}$; {The $i$th city.}[a]
**end for**
{Verification:}
**if** $x_1, x_2, \ldots, x_n$ are distinct and $\sum_{i=1}^{n-1} d_{x_i, x_{i+1}} \leq B$ **then**
  "yes";
**else**
  "no";
**end if**

[a]Can be made into a series of $\log_2 n$ *binary* choices for each $x_i$ so that the next-state count (2) is a constant, independent of input size. Contributed by Mr. Chih-Duo Hong (R95922079) on September 27, 2006.

## Analysis

- Suppose the input graph contains at least one tour of the cities with a total distance at most $B$.
  - Then there is a computation path for that tour.[a]
  - And it leads to "yes."
- Suppose the input graph contains no tour of the cities with a total distance at most $B$.
  - Then every computation path leads to "no."

[a]It does not mean the algorithm will follow that path. It merely means such a computation path (i.e., a sequence of nondeterministic choices) exists.

## Time Complexity under Nondeterminism

- Nondeterministic machine $N$ decides $L$ **in time** $f(n)$, where $f : \mathbb{N} \to \mathbb{N}$, if
  - $N$ decides $L$, and
  - for any $x \in \Sigma^*$, $N$ does not have a computation path longer than $f(|x|)$.
- We charge only the "depth" of the computation tree.

## Time Complexity Classes under Nondeterminism

- NTIME($f(n)$) is the set of languages decided by NTMs within time $f(n)$.
- NTIME($f(n)$) is a complexity class.

## NP ("Nondeterministic Polynomial")

- Define
$$\text{NP} \stackrel{\Delta}{=} \bigcup_{k>0} \text{NTIME}(n^k).$$
- Clearly $\text{P} \subseteq \text{NP}$.
- Think of NP as efficiently *verifiable* problems (see p. 343).
  - Boolean satisfiability (p. 120 and p. 201), e.g.
- The most important open problem in computer science is whether $\text{P} = \text{NP}$.

# Remarks on the $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$ Open Problem[a]

- Many practical applications depend on answers to the $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$ question.
- Verification of password should be easy (so it is in NP).
  - A computer should not take a long time to let a user log in.
- A password system should be hard to crack (loosely speaking, cracking it should not be in P).
- It took 63 years to settle the Continuum Hypothesis; how long will it take for this one?

[a] Contributed by Mr. Kuan-Lin Huang (`B96902079`, `R00922018`) on September 27, 2011.

## Simulating Nondeterministic TMs

Nondeterminism does not add power to TMs.

**Theorem 6** *Suppose language $L$ is decided by an NTM $N$ in time $f(n)$. Then it is decided by a 3-string deterministic TM $M$ in time $O(c^{f(n)})$, where $c > 1$ is some constant depending on $N$.*

- On input $x$, $M$ explores the computation tree of $N(x)$ using depth-first search.
  - $M$ does *not* need to know $f(n)$.
  - As $N$ is time-bounded, the depth-first search will halt.

# The Proof (concluded)

- If any path leads to "yes," then $M$ immediately enters the "yes" state.
- If none of the paths lead to "yes," then $M$ enters the "no" state.
- The simulation takes time $O(c^{f(n)})$ for some $c > 1$ because the computation tree has that many nodes.

**Corollary 7** $\text{NTIME}(f(n)) \subseteq \bigcup_{c>1} \text{TIME}(c^{f(n)})$.[a]

[a]Mr. Kai-Yuan Hou (B99201038, R03922014) on October 6, 2015: $\bigcup_{c>1} \text{TIME}(c^{f(n)}) \subseteq \text{NTIME}(f(n))$?

# NTIME vs. TIME

- Does converting an NTM into a TM *require* exploring all computation paths of the NTM in the worst case as done in Theorem 6 (p. 132)?
- This is a key question in theory with important practical implications.

# Nondeterministic Space Complexity Classes

- Let $L$ be a language.
- Then
$$L \in \text{NSPACE}(f(n))$$
if there is an NTM with input and output that decides $L$ and operates within space bound $f(n)$.
- $\text{NSPACE}(f(n))$ is a set of languages.
- As in the linear speedup theorem,[a] constant coefficients do not matter.

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[a]Theorem 5 (p. 96).

## Graph Reachability

- Let $G(V, E)$ be a directed graph (**digraph**).
- REACHABILITY asks, given nodes $a$ and $b$, does $G$ contain a path from $a$ to $b$?
- Can be easily solved in polynomial time by breadth-first search.
- How about its *nondeterministic* space complexity?

## The First Try: NSPACE($n \log n$)

Determine the number of nodes $m$; {Note $m \leq n$.}
$x_1 := a$; {Assume $a \neq b$.}
**for** $i = 2, 3, \ldots, m$ **do**
  Guess $x_i \in \{ v_1, v_2, \ldots, v_m \}$; {The $i$th node.}
**end for**
**for** $i = 2, 3, \ldots, m$ **do**
  **if** $(x_{i-1}, x_i) \notin E$ **then**
    "no";
  **end if**
  **if** $x_i = b$ **then**
    "yes";
  **end if**
**end for**
"no";

# In Fact, REACHABILITY $\in$ NSPACE$(\log n)$

```
Determine the number of nodes m; {Note m ≤ n.}
x := a;
for i = 2, 3, ..., m do
    Guess y ∈ { v_1, v_2, ..., v_m }; {The next node.}
    if (x, y) ∉ E then
        "no";
    end if
    if y = b then
        "yes";
    end if
    x := y; {Recycle the space.}
end for
"no";
```

## Space Analysis

- Variables $m$, $i$, $x$, and $y$ each require $O(\log n)$ bits.
- Testing $(x, y) \in E$ is accomplished by consulting the input string with counters of $O(\log n)$ bits long.
- Hence

$$\text{REACHABILITY} \in \text{NSPACE}(\log n).$$

  - REACHABILITY with more than one terminal node also has the same complexity.
  - In fact, REACHABILITY for *undirected* graphs is in SPACE($\log n$).[a]
- It is well-known that REACHABILITY $\in$ P.[b]

[a] Reingold (2004).

[b] See, e.g., p. 246.

*Undecidability*

He [Turing] invented
the idea of software, essentially[.]
It's software that's really
the important invention.
— Freeman Dyson (2015)

# Universal Turing Machine[a]

- A **universal Turing machine** $U$ interprets the input as the *description* of a TM $M$ concatenated with the *description* of an input to that machine, $x$.[b]
  - Both $M$ and $x$ are over the alphabet of $U$.
- $U$ simulates $M$ on $x$ so that
$$U(M; x) = M(x).$$
- $U$ is like a modern computer, which executes any valid machine code, or a Java virtual machine, which executes any valid bytecode.

[a]Turing (1936) calls it "universal computing machine."
[b]See pp. 57–58 of the textbook.

# The Halting Problem

- **Undecidable problems** are problems that have no algorithms.
  - Equivalently, they are languages that are not recursive.
- We now define a concrete undecidable problem, the **halting problem**:

$$H \stackrel{\Delta}{=} \{ M; x : M(x) \neq \nearrow \}.$$

  - Does $M$ halt on input $x$?
- $H$ is called the **halting set**.

# $H$ Is Recursively Enumerable

- Use the universal TM $U$ to simulate $M$ on $x$.
- When $M$ is about to halt, $U$ enters a "yes" state.
- If $M(x)$ diverges, so does $U$.
- This TM accepts $H$.

## $H$ Is Not Recursive[a]

- Suppose $H$ is recursive.
- Then there is a TM $M_H$ that *decides* $H$.
- Consider the program $D(M)$ that calls $M_H$:

```
if M_H(M; M) = "yes" then
    ↗; {Inserting an infinite loop here.}
else
    "yes";
end if
```

---

[a] Turing (1936).

## $H$ Is Not Recursive (concluded)

- Consider $D(D)$:
  - $D(D) = \nearrow \Rightarrow M_H(D; D) =$ "yes" $\Rightarrow D; D \in H \Rightarrow D(D) \neq \nearrow$, a contradiction.
  - $D(D) =$ "yes" $\Rightarrow M_H(D; D) =$ "no" $\Rightarrow D; D \notin H \Rightarrow D(D) = \nearrow$, a contradiction.

## Comments

- Two levels of interpretations of $M$:[a]
  - A sequence of 0s and 1s (data).
  - An encoding of instructions (programs).
- There are no paradoxes with $D(D)$.
  - Concepts should be familiar to computer scientists.
  - Feed a C compiler to a C compiler, a Lisp interpreter to a Lisp interpreter, a sorting program to a sorting program, etc.

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[a] Eckert & Mauchly (1943); von Neumann (1945); Turing (1946).

It seemed unworthy of a grown man
to spend his time on such trivialities,
but what was I to do? [· · ·]
The whole of the rest of my life might be
consumed in looking at
that blank sheet of paper.
— Bertrand Russell (1872–1970),
*Autobiography*, Vol. I (1967)

## Self-Loop Paradoxes[a]

**Russell's Paradox (1901):** Consider $R = \{A : A \notin A\}$.

- If $R \in R$, then $R \notin R$ by definition.
- If $R \notin R$, then $R \in R$ also by definition.
- In either case, we have a "contradiction."[b]

**Epimenides and Eubulides:** The Cretan says, "All Cretans are liars."[c]

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[a] E.g., Quine (1966), *The Ways of Paradox and Other Essays* and Hofstadter (1979), *Gödel, Escher, Bach: An Eternal Golden Braid*.

[b] Gottlob Frege (1848–1925) to Bertrand Russell in 1902, "Your discovery of the contradiction [. . .] has shaken the basis on which I intended to build arithmetic."

[c] Also quoted in *Titus* 1:12.

## Self-Loop Paradoxes (continued)

**Liar's Paradox:** "This sentence is false."

**Hypochondriac:** a patient with imaginary symptoms and ailments.[a]

**Sharon Stone in *The Specialist* (1994):** "I'm not a woman you can trust."

***Numbers* 12:3:** "Moses was the most humble person in all the world [$\cdots$]" (attributed to Moses).

***Psalms* 116:11:** "Everyone is a liar."

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[a] Like Gödel and the pianist Glenn Gould (1932–1982).

# Self-Loop Paradoxes (continued)

**A restaurant in Boston:** No Name Restaurant (1917–2020).

**U.S. Department of State (March 19, 2020):** U.S. citizens who live in the United States should arrange for immediate return to the United States[.]

***The Egyptian Book of the Dead*:** "ye live in me and I would live in you."[a]

[a] See also *John* 14:10 and 17:21.

## Self-Loop Paradoxes (concluded)

**Jerome K. Jerome (1887), *Three Men in a Boat*:** "How could I wake you, when you didn't wake me?"

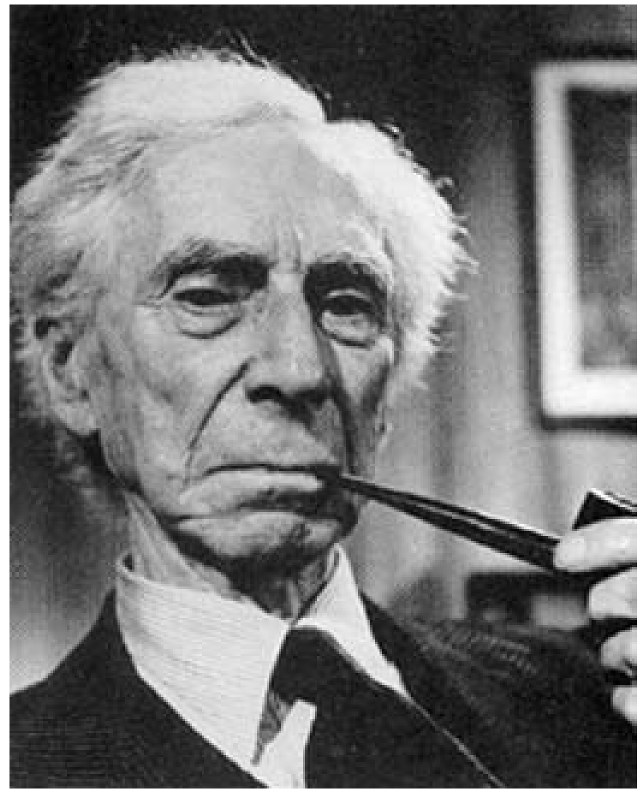
**Winston Churchill (January 23, 1948):** "For my part, I consider that it will be found much better by all parties to leave the past to history, especially as I propose to write that history myself."

**Nicola Lacey (2004), *A Life of H. L. A. Hart*:** "Top Secret [MI5] Documents: Burn before Reading!"

# Bertrand Russell[a] (1872–1970)

Norbert Wiener (1953), "It is impossible to describe Bertrand Russell except by saying that he looks like the Mad Hatter."

Karl Popper (1974), "perhaps the greatest philosopher since Kant."



[a]Nobel Prize in Literature (1950).

## Reductions in Proving Undecidability

- Suppose we are asked to prove that $L$ is undecidable.
- Suppose $L'$ (such as $H$) is known to be undecidable.
- Find a computable transformation $R$ (called **reduction**[a]) from $L'$ to $L$ such that[b]

$$\forall x \, \{ \, x \in L' \text{ if and only if } R(x) \in L \, \}.$$

- Now we can answer "$x \in L'$?" for *any* $x$ by answering "$R(x) \in L$?" because it has the same answer.

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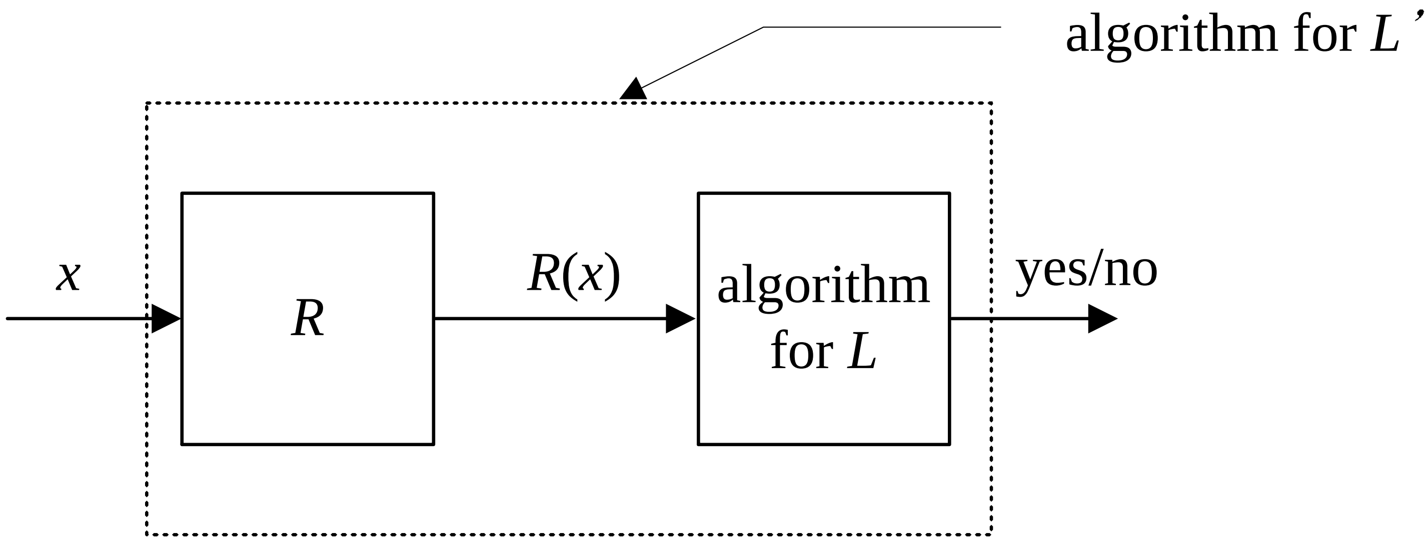
[a] Post (1944).

[b] Contributed by Mr. Tai-Dai Chou (J93922005) on May 19, 2005.


algorithm for L'
x
R
R(x)
algorithm
for L
yes/no

## Reductions in Proving Undecidability (concluded)

- $L'$ is said to be **reduced** to $L$.[a]
- If $L$ were decidable, "$R(x) \in L$?" becomes computable and we have an algorithm to decide $L'$, a contradiction!
- So $L$ must be undecidable.

**Theorem 8** *Suppose language $L_1$ can be reduced to language $L_2$. If $L_1$ is undecidable, then $L_2$ is undecidable.*

[a] Intuitively, $L$ can be used to solve $L'$.

## Special Cases and Reduction

- Suppose $L_1$ can be reduced to $L_2$.
- As the reduction $R$ maps members of $L_1$ to a *subset* of $L_2$,[a] we *may* say $L_1$ is a "special case" of $L_2$.[b]
- That is one way to understand the use of the somewhat confusing term "reduction."

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[a] Because $R$ may not be onto.

[b] Contributed by Ms. Mei-Chih Chang (`D03922022`) and Mr. Kai-Yuan Hou (`B99201038`, `R03922014`) on October 13, 2015.

## Subsets and Decidability

- Suppose $L_1$ is undecidable and $L_1 \subseteq L_2$.
- Is $L_2$ undecidable?[a]
- It depends.
- When $L_2 = \Sigma^*$, $L_2$ is decidable: Just answer "yes."
- If $L_2 - L_1$ is decidable, then $L_2$ is undecidable.
  - Clearly,

$$x \in L_1 \text{ if and only if } x \in L_2 \text{ and } x \notin L_2 - L_1.$$

  - Therefore, if $L_2$ were decidable, then $L_1$ would be.

[a] Contributed by Ms. Mei-Chih Chang (D03922022) on October 13, 2015.

## Subsets and Decidability (concluded)

- Suppose $L_2$ is decidable and $L_1 \subseteq L_2$.
- Is $L_1$ decidable?
- It depends again.
- When $L_1 = \emptyset$, $L_1$ is decidable: Just answer "no."
- But if $L_2 = \Sigma^*$ and $L_1 = H$, then $L_1$ is undecidable.

## The Universal Halting Problem

- The **universal halting problem**:

$$H^* \stackrel{\Delta}{=} \{ M : M \text{ halts on all inputs} \}.$$

- It is also called the **totality problem**.

## $H^*$ Is Not Recursive[a]

- We will reduce $H$ to $H^*$.
- Given the question "$M; x \in H$?", construct the following machine (this is the reduction):[b]

$$M_x(y) \; \{ M(x); \}$$

- $M$ halts on $x$ if and only if $M_x$ halts on all inputs.
- In other words, $M; x \in H$ if and only if $M_x \in H^*$.
- So if $H^*$ were recursive (recall the box for $L$ on p. 155), $H$ would be recursive, a contradiction.

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[a]Kleene (1936).

[b]Simplified by Mr. Chih-Hung Hsieh (D95922003) on October 5, 2006. $M_x$ ignores its input $y$; $x$ is part of $M_x$'s code but not $M_x$'s input.

## More Undecidability

- $\{ M; x : \text{there is a } y \text{ such that } M(x) = y \}$.
- $\{ M; x :$
  the computation $M$ on input $x$ uses all states of $M \}$.
- $\{ M; x; y : M(x) = y \}$.